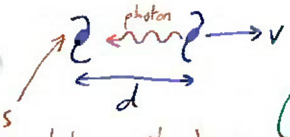
G. H. and C. ①

let $v = \frac{d}{\text{age of universe}} = dH_0$

Main Features of Universe



we see photons with λ_{obs}:

REDSHIFT $z = \frac{\lambda_{obs} - \lambda_{emitted}}{\lambda_{emitted}}$

$$\boxed{z = \frac{\lambda_o}{\lambda_e} - 1}$$

Hubble's Law:

$$\boxed{v = H(t)\,d}$$

$(H(t_0) = H_0)$ ← present.

if v is non rel, $\nu_{obs} = \nu_{em}(1 - \frac{v}{c})$

$\Rightarrow zc \simeq H_0 d$

$H_0 \simeq 50\text{-}80\ \text{kms}^{-1}\text{Mpc}^{-1}$

So age of univ $\sim \frac{1}{H_0} = 12.5 \to 20$ billion yrs.

CMBR

v. good BBR

T = 2.74 K

⇒ universe at one time must have been in T.E.

Olber's Paradox

each object: Luminosity L

So at centre of sphere get

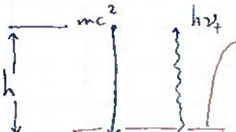
$\frac{L}{4\pi r^2} \times \underbrace{4\pi r^2 dr\,\rho}_{\text{number of objects in shell}}$

So total flux $= \int_0^\infty L\rho\,dr \to \infty$ in infinite universe

actually stars shield etc.... so instead, every line of sight ends on star....

⇒ $\boxed{\text{universe is expanding}}$

Einstein's Tower



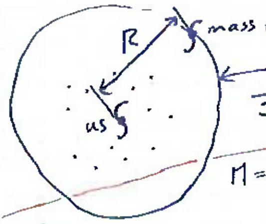
$\boxed{mc^2 + mgh = h\nu_b}$

So $h\nu_b = mc^2(1 + \frac{gh}{c^2})$

$\Rightarrow \nu_b = \nu_t(1 + \frac{gh}{c^2})$

$$\Rightarrow \boxed{z = \frac{gh}{c^2} = \frac{\Delta\emptyset}{c^2}}$$

Application of Newtonian Gravity



Gauss ⇒ only consider mass inside sphere.

in sphere ∃ mass M

$M = \frac{4\pi}{3}R^3\rho$

$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$

but $v = HR$ so

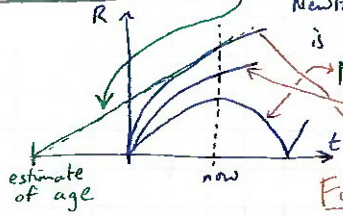
$$\boxed{E = \frac{1}{2}mR^2H^2\Big(1 - \underbrace{\frac{8\pi G\rho}{3H^2}}_{\Omega}\Big)}$$

Also, using $HR = \dot{R}$

get $\boxed{\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} = \frac{2E}{mR^2}}$ $\left(= -\frac{kc^2}{R^2} \text{ in GR}\right)$

So let $\frac{E}{m} = -kc^2$ and rescale R

ie rescaling R means Newtonian "radial distance" is not same as GR "scale factor".



Now $k = +1$ ie $\Omega > 1 \Rightarrow E\ -^{ve}$ ∴ univ bound

$k = 0$ ie $\Omega = 0 \Rightarrow E = 0$ ∴ critical

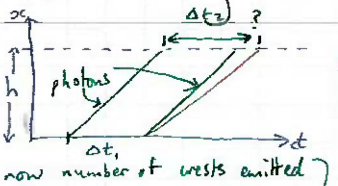
$k = -1$ ie $\Omega < 1 \Rightarrow E\ +^{ve}$ ∴ unbound

Force eq'n: $\frac{GMm}{R^2} = -m\ddot{R} \Rightarrow \boxed{\underbrace{\frac{\ddot{R}}{R}}_{-ve} + \underbrace{\frac{4\pi G\rho}{3}}_{>0} = 0}$

⇒ Static univ. not possible!

also, $\ddot{R}$ is $-^{ve}$ so expansion is slowing down.

Gravity and Special Relativity

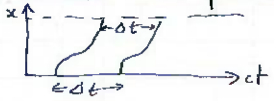


now number of waves emitted $= N = \Delta t_1 \nu_1 = \Delta t_2 \nu_2$

but $\nu_1 > \nu_2$ ∴ $\Delta t_2 > \Delta t_1$

but on space-time diag, photons must travel at 45° - red line is not allowed.

This is true no matter how gravity alters photon paths through spacetime because the pattern is time-invariant ie.



So: Incompatible

[NB - phase is the invariant]

Principles of Equivalence

STRONG: At any pt in grav. field in a free fall Lorentz frame all the laws of physics have their usual S.R. form.

WEAK: At any pt in grav field in a free fall Lorentz frame the laws of motion have their usual S.R. form

except grav - disappears of free test particles ie straight lines through space

these two are equivalent to this

$\Rightarrow \boxed{m_{inertial} = m_{grav}}$

ie $\frac{GMm_g}{r^2} = m_i\,\text{acc.}$ - take $\frac{m_i}{m_g} = 1$ usually

We think this is true because:

① Galileo - tower of Piza + inclined planes - 1 part in 10^2

② Eötvos 1922 - torsion balance - 1 p.p. 10^9

③ 1964, 71 ... 3 arm torsion balance 1 pp 10^{12}

also sub atomic particles.

ie. motion of body is indep. of composition - only dep on pos'n + vel...

can prove for e/m for tower using doppler shift

Schwarzschild Metric - time part

$z = \frac{T_o}{T_e} - 1 = \frac{\Delta\emptyset}{c^2} = \frac{GM}{Rc^2}$ from body mass M...

T_o ← dt - observed time

T_e ← ds - proper time

So $\left(\frac{ds}{dt}\right)^2 = \frac{1}{\left(1 + \frac{GM}{Rc^2}\right)^2} \Rightarrow ds^2 = \left(1 - \frac{2GM}{Rc^2}\right)dt^2$

Binomial expansion - needed to approximate as z formula is only first order. tested using atomic clocks.

(sit on BH stay young)

Schwarszschild Metric - radial part

require ISOTROPY ⇒ angular part must be undistorted (flat).

then use sym and work in the equatorial plane: $ds^2 = f(r)dr^2 + r^2(d\theta^2_0 + \sin^2(\frac{\pi}{2})d\emptyset^2)$ (~~so can use this result~~ not yes)

Use dimensions: K(r) only f'n of M, r, G, c

guess ∝ M then get $K(r) = \alpha \frac{GM}{c^2r^3}$ ($\alpha = -1$ from GR or weak limit)

then use $K = \frac{f'}{2f^2r}$

to get $-\frac{1}{f} = \frac{2GM}{rc^2}$ + const another

then get:

$$\boxed{ds^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - \frac{r^2d\theta^2}{c^2} - \frac{r^2\sin^2\theta\,d\emptyset^2}{c^2}}$$

SCHWARZSCHILD METRIC

notice for $r < \frac{2GM}{c^2}$, a t change is space-like and an r change is time-like - they swap roles.

Metric for Constant curvature

2-D surface, $(r, \emptyset)$

metric: $g = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix}$

plug into Gauss's beautiful theorem

get $\boxed{K = \frac{f'}{2f^2r}}$ if require K is constant

then get: $\boxed{f(r) = \frac{1}{1 - Kr^2}}$

Geodesics

$S_{AB} = \int_A^B G(x^\mu, \dot{x}^\mu)\,ds$

where $G = (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2} = 1$ by construction. constant!

$$\Rightarrow \frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu}$$

THE GEODESIC EQUATION.

Curvature must exist in space-time - there is only one Geodesic through each point. ie. different direction in space-time means different initial speed (eg cannon ball + tennis ball)

G.A. and C. ②

Schwarzschild Geodesics

We have $G^2 = \alpha \dot{t}^2 - \frac{\dot{r}^2}{c^2 \alpha} - \frac{r^2 \dot{\phi}^2}{c^2}$

$$\boxed{\alpha = \left(1 - \frac{2GM}{rc^2}\right)}$$

Work in $\theta = \frac{\pi}{2}$ plane – OK. coz cons. ang mom.

t eq'n: $\alpha \dot{t} = k$ conserved

ϕ eq'n: $\underline{r^2 \dot{\phi} = h}$ use in here to get

r eq'n gives: (actual just set $G = 1$ in here)

$$\boxed{\dot{r}^2 = c^2(k^2-1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}}$$

$$\boxed{\underbrace{\frac{1}{2}m\dot{r}^2}_{\text{NEWT.}} + \underbrace{\frac{1}{2}m(r\dot{\phi})^2 \alpha}_{\text{NEWT. almost}} - \underbrace{\frac{GMm}{r}}_{\text{newt.}} = \frac{1}{2}mc^2(k^2-1)}$$

energy eq'n. $k = \frac{E}{mc^2}$

EULER-LAGRANGE EQ'NS

The substitution $u = \frac{1}{r}$ changes ellipse eq'n to trig form.

(diff. w.r.t. ϕ removes const)

SHAPE OF ORBIT:

Want $r(\phi)$: $\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{h}{r^2}\frac{dr}{d\phi}$

$$\Rightarrow \left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = c^2(k^2-1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2 r^3}$$

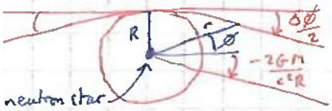
$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2(k^2-1)}{h^2} + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2} \quad \left(u = \frac{1}{r}\right)$$

$$\Rightarrow \boxed{\underbrace{\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}}_{\text{NEWTON}} + \underbrace{\frac{3GMu^2}{c^2}}_{\text{EINSTEIN}}}$$ usual orbit eq'n

Light bending

For photons $d\tau = 0 \therefore h = \infty$ (as $r^2 \frac{d\phi}{d\tau} = h$)

So there is no newtonian term



ie $\boxed{\frac{d^2u}{d\phi^2} + u = \frac{3GMu^2}{c^2}}$ for photons.

now – hom. eq'n is satisfied by $u = \frac{\sin\phi}{R}$ – the undisturbed track

then for inhom eq'n set $u^2 = \frac{\sin^2\phi}{R^2}$ to get approx. solution:

$$\boxed{u = \frac{\sin\phi}{R} + \frac{3GM}{2c^2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right)}$$

let $r \to \infty$ ie $u \to 0$, ϕ small,

$$\Rightarrow \boxed{\Delta\phi = \frac{4GM}{c^2 R}}$$

Circular Orbits

Need energy of circular orbit.

put $u = $ const to get h then use energy eq'n with $\dot{r} = 0$ and $r^2\dot{\phi} = h$

Then:

$$\boxed{E = \frac{mc^2\left(1 - \frac{2GM}{rc^2}\right)}{\left(1 - \frac{3GM}{rc^2}\right)^{1/2}}}$$

expand for $r \to \infty$ get $E \simeq \underset{\text{at }\infty}{mc^2} - \underbrace{\frac{GMm}{2r}}_{(<\text{ energy at }\infty)} + \ldots$

cf. Newton:

$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2r}$

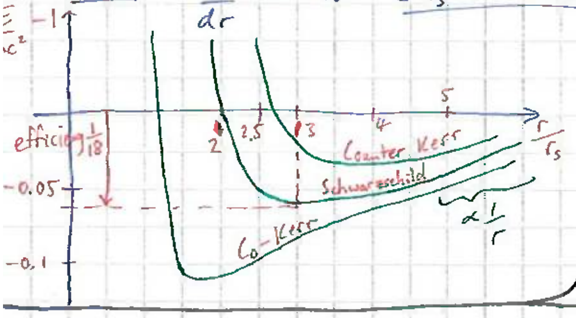
same – so Newtonian energy is a correction to rest mass energy

Photon orbits: $m \to 0$ then $\boxed{r = \frac{3GM}{c^2}}$ Photon orbit radius

can cancel zero

For $E < 0$, orbits are bound: $2r_s < r < \infty$

but stable? $\frac{dE}{dr} > 0$ for $3r_s < r < \infty$



Non-Circular Orbits - Capture

Take energy eq'n - subst for $\dot{\phi}$ then diff. w.r.t. to τ then elim $\dot{r}$ using orig. equation.

get: $\ddot{r} = -\frac{GM}{r} + \underbrace{\frac{h^2}{r^3}}_{} - \underbrace{\frac{3h^2GM}{c^2r^4}}_{}$ GR term

centrifugal force $\frac{r^4\dot{\phi}^2}{r^3}$

For $r = \frac{3GM}{c^2}$ and less, GR term is > centr. term – unstable to collapse.

eg

FRW Metric

for each time.

Require spatial sections to be isotropic (and homogenous

$\Rightarrow ds^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

must yield constant curvature K

Take $\theta = \frac{\pi}{2}$ then $\boxed{f = \frac{1}{1-Kr^2}}$

(OK coz can rotate)

3 cases:
- $K = 0$ – flat, euclidean space.
- $K > 0$ $\Delta s = \int_0^{\Delta s} ds = \int_0^r \frac{dr}{\sqrt{1-Kr^2}} = \frac{1}{\sqrt{K}}\sin^{-1}(r\sqrt{K})$

same; at const ϕ

So $r = \frac{\sin(a\sqrt{K})}{\sqrt{K}}$ — CLOSED UNIVERSE.

coord. dist. proper dist.

- $K < 0$ $r = \frac{1}{\sqrt{-K}}\sinh^{-1}(a\sqrt{K})$

So $r = \frac{1}{\sqrt{-K}}\sinh(a\sqrt{-K})$ — OPEN UNIVERSE.

Time part:

Only way to preserve hom./is is $K = \boxed{K(t) = \frac{k}{R^2(t)}}$ $k = -1, 0, 1$

let $r = \sigma R$

then $\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ becomes $R^2\underbrace{\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right]}_{\text{fixed 3-metric}}$

overall scaling

So the proper distance for $\boxed{\text{fixed } \sigma}$ increases with $R(t)$

But: we want t such that in a certain global frame of rest, everyone can measure curvature (etc etc). Is there such a frame? Yes – objects with fixed σ are in it – called fundamental observers (eg galaxies with no peculiar motion) (or comoving obs.)

f.o.'s have no CMBR dipole anisotropy.

Cosmic time is proper time measured by comoving observers.

and
$$\boxed{ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right]}$$

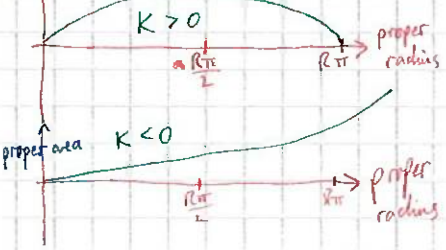
Comoving Observers follow Geodesics of this metric.

Proper area of surface of a sphere with proper radius a is $4\pi r^2$

So for $K > 0$, Area $A = \frac{4\pi}{K}\sin^2(a\sqrt{K})$

for $K < 0$ $A = \frac{4\pi}{|K|}\sinh^2(a\sqrt{|K|})$

ie. do Euclidean Geometry with coordinate variables then relate to actual measured (proper) variables using metric!



G.H. and C (3)

Stellar Structure Equations

$t_{\text{free fall}} = \frac{\text{radius}}{\text{free fall vel.}} = \underbrace{\sqrt{\frac{R^3}{GM}}}_{27 \text{ min for } \odot} \ll t_{\text{kelvin}} = \frac{\text{grav. b.e.}}{L} = \frac{GM}{RL} \sim 10$ yrs for ⊙

ie collapses in 30 mins to radius where luminosity keeps it stable - then stays there.

Hydrostatic equilibrium

$$\Rightarrow \boxed{\frac{dp}{dr} = -\rho \frac{GM(r)}{r^2}} ①$$

where $\boxed{M(r) = \int_1^r \rho(r') 4\pi r'^2 dr'}$ ② (def'n of M, ρ)

Can estimate central temp of star using ideal gas law: $T = \frac{pV}{nk} = \frac{GM^2}{R^4} \cdot \frac{\mu m}{k\rho} \sim 10^7$ K $\underline{0 < \mu < ?}$ ← is just a number

$\Rightarrow \underset{\text{K.E.}}{kT} \sim \frac{GM(\mu m)}{\underset{\text{P.E.}}{}R}$ - a special case of the Virial Theorem.

Energy Conservation $\Rightarrow \boxed{\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon}$ ③ where ε is the rate of energy generation.

Energy Transfer $\Rightarrow \boxed{\underbrace{\frac{L}{4\pi r^2}}_{\text{flux}} = -\frac{4acT^3}{3\kappa\rho} \underbrace{\frac{dT}{dr}}_{\text{temp gradient.}}}$ ④ $\propto$

— because: for radiative transport,

$F = -\underset{\text{diffusion coefficient.}}{D} \frac{dU}{dr}$ ← energy density in radiation at r ie aT^4

but $D = \underbrace{\frac{\lambda c}{3}}_{\text{Kinetic theory}} = \frac{c}{3} \cdot \frac{1}{n\sigma} = \frac{c}{3} \frac{1}{\kappa\rho}$ ie. $\underline{\sigma = \kappa \mu m}$

Virial Theorem

For a cluster of inverse square law point charges, $\boxed{\langle E_{kin} \rangle = -\frac{\langle E_{pot} \rangle}{2}}$

K.E. must be < P.E. otherwise unbound!

Now $E_{pot} = -\int_0^R \frac{GM(r)}{r} \rho 4\pi r^2 dr$

← bring shell from ∞

$= \int_0^R \frac{dp}{dr} 4\pi r^3 dr$

$= \cancel{\left[p \, 4\pi r^3 \right]_0^R}^{\,0} - \int_0^R p\, dV$

Now for gas $p = nkT$, $\boxed{E_k = \frac{3}{2}nkT}$ hence

(for radiation, get $E_k = -E_p$)

True only for points! Generally $E_k = \frac{nfkT}{2} = \frac{nkT}{\gamma - 1}$

$\Rightarrow \boxed{\langle E_{kin} \rangle = \frac{-\langle E_{pot} \rangle}{3(\gamma - 1)}}$ So total $E = E_p \frac{(3\gamma - 4)}{3(\gamma - 1)}$

So unbound for $\gamma < \frac{4}{3}$ $\left(\gamma = \frac{f+2}{f}\right)$

ie. push in cloud ← photons f = 6

- if ~~no~~ f = 3 pressure pushes out.

if f > 3, work done → rot. etc no can push out so collapse (or explode).

Energy Generation

This is an equilibrium situation

Before eq. is reached, have gas cloud, collapses: grav. energy → high temp

$\Rightarrow$ Fusion can occur. need $T \sim 10^7$ K

(N.B. QM tunneling process ∴ steep T dependence)

p-p chain:

$p + p \rightarrow d + e^+ + \nu_e$

$p + d \rightarrow He^3 + \gamma$

$He^3 + He^3 \rightarrow He^4 + 2p$

energy released is $\sim 0.007 \times 4 m_p c^2$

$\varepsilon \propto \rho T^4$

In very massive stars, the CNO cycle is important $(\varepsilon \propto \rho T^{15})$

Stellar Evolution

: Solve eq'ns ① → ④ and an eq'n for κ - opacity and all tha

Note: $\underbrace{T \propto \frac{M}{R}, \quad \frac{L}{R^2} \propto \frac{T^4}{\kappa\rho R}}_{\text{scaling laws.}}$

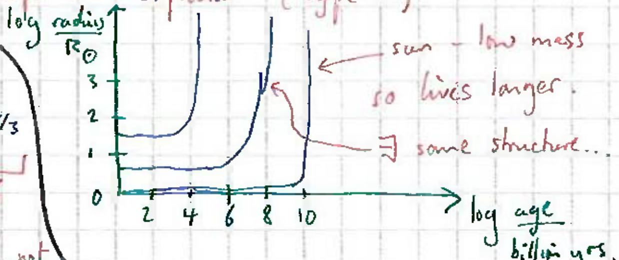
Hydrogen burning lasts $\sim 10^{10}$ yrs. When use all H, core contracts until hot enough for He burning whilst outer part swells by ~100 times or more - red giant phase.

Massive stars: Radiation pressure dominates.

When? : $P_g \sim P_{rad} \Rightarrow nkT \sim aT^4$ or, $\frac{GM^2}{R^4} \sim \frac{aG^4M^4(\mu m)}{R^4k^4}$

So for $M \gtrsim 100 M_\odot$ the star is unstable - Supernova explosion (Type II)

~~ie~~ so can get limit on $M \simeq 100 M_\odot$



sun - low mass so lives longer.

∃ some structure...

White Dwarfs

DEGENERACY PRESSURE:

Estimate via $\Delta x \Delta p_x \sim \hbar$

use $P_{deg} = \frac{E_{deg} N}{V}$, $n = \frac{N}{V}$, $\Delta x \sim \left(\frac{V}{N}\right)^{1/3}$

Non rel limit: $E_{deg} = \frac{p^2}{2m} \Rightarrow P^{n.r.}_{deg} \simeq \frac{\hbar^2}{2m} n^{5/3}$

Rel limit: $E_{deg} = pc \Rightarrow P^{rel}_{deg} \simeq \hbar c \, n^{4/3}$

c.f. ideal gas $P \propto \rho^\gamma$

so non rel stable, rel, not.

Aside: lower mass limit on stars:

for p-p chain need $T \sim \frac{GMm}{Rk} \sim 10^7$ K. ie, need $P_{thermal} > P_{deg}$

$\Rightarrow$ need Mass $\gtrsim 0.005 M_\odot$ ie $\frac{M_\odot}{20}$

For $M < \frac{M_\odot}{20}$ deg. pressure supports. For $M < \frac{M_\odot}{1000}$ get electrostatic + bit of deg. pressure eg earth

$P_{deg} \propto \frac{1}{\text{mass of degenerate particle}}$ ∴ electron effect >> proton etc effect.

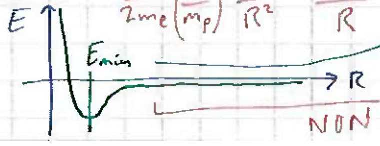
Objects supported by e^- deg. pressure are WHITE DWARFS

Total energy of W.D. ≃ $NE_{deg} + E_{grav}$

$\simeq \frac{\hbar^2}{2m_e} \cdot \frac{N^{5/3}}{V^{2/3}} - \frac{GM^2}{R}$

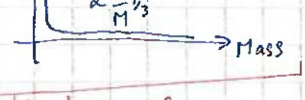
but $N \simeq \frac{M}{m_p}$ and $V \simeq R^3$

So $E \simeq \frac{\hbar^2}{2m_e} \left(\frac{M}{m_p}\right)^{5/3} \frac{1}{R^2} - \frac{GM^2}{R}$



→ occurs at $M \simeq \left(\frac{\hbar^6}{m_e^3 G^3 m_p^5}\right) \frac{1}{R^3}$

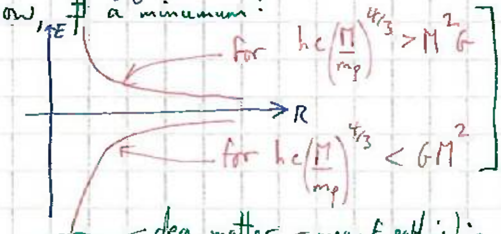
So radius decreases as the mass increases



NON-relativistic case

Total energy of a relativistic W.D. is $\hbar c \left(\frac{M}{m_p}\right)^{4/3} \frac{1}{R} - \frac{GM^2}{R}$

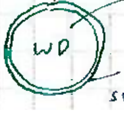
now, ∄ a minimum:



cross over point, get Chandrasekar mass

$$\boxed{M_{Ch} = \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2}}$$

ie mass where instability changes from explode to collapse → B.H.



deg. matter - mean f. path is big ∴ efficient heat conduction

non deg blanket - keeps heat in so cool in 10^{10} yrs - hot + white.

G. A. and C. ④

Neutron Stars

- Form when $E_{deg} > (m_n - m_p)c^2$ (1.29 MeV)

Then can get inverse β-decay:

$e + p \rightleftarrows n + \nu$

reverse process blocked by lack of free e^- states.

Occurs at $\sim 10^{10}\,\mathrm{kg\,m^{-3}}$. As $\rho \uparrow$, get neutron degeneracy

For W.D. $M \propto \dfrac{1}{R^3 \times (\text{mass of deg particle})^3}$

So can use to get $R_{ns} \simeq \underbrace{R_{wd}}_{R_\oplus} \left(\dfrac{m_e}{m_n}\right) \sim 10\,\mathrm{km}$

Interparticle separation:

$\Delta x \sim \left(\dfrac{N}{V}\right)^{-1/3} \sim 10^{-15}\,\mathrm{m}$ - same as nuc. radius

ie. the nuclei are "touching" - it is one gigantic nucleus.

R_{ns} is $O(R_s)$ so need GR...

$\underbrace{0.1 M_\odot < M < 3M_\odot}$

or else ∃ free e^- states so β decay occurs → W.D.

Supernovae

For $M > 8M_\odot$ (otherwise WD forms)

Red Giant: fusion → Iron.

Uses up fuel then core collapses → the collapse releases grav. energy

- neutrons: captured by outer envelope making iron etc
- γ-photons: photo-dissociate Fe core into n, p ...
- neutrinos: v. high densities so diffusion of ν slows collapse from t_{ff} to a few seconds (10s)

Neutrino luminosity $\sim \dfrac{BE}{t_{diff}} \sim 10^{45}\,\mathrm{W}$

Then the core "bounces" - outer envelope is emitted at $10^4\,\mathrm{km\,s^{-1}}$

Luminosity $\sim 10^9 L_\odot$

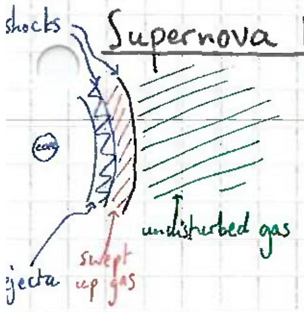
core forms n.s. or b.h. for $M > 30M_\odot$

TYPE I Supernovae ≠ TYPE II Supernovae

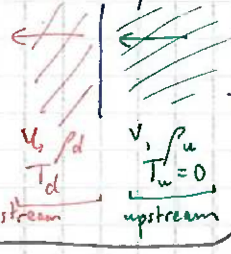
W.D. in binary sys. accretes s.t. $M > M_{Ch}$ then just blows up - there is no remaining compact object.

EXAMPLE: SN 1987A About 20 ν were detected within about 20s - (can get m_ν limit)

Supernova Remnants



In rest frame of outer shock



Mass cons: $\rho_u V = \rho_d u$

Momentum conservation: $\rho_u V^2 = P_d + \rho_d u^2$

Energy: $V\left(\frac{1}{2}\rho_u V^2\right) - u\left(\frac{1}{2}\rho_d u^2 + \frac{3}{2}P_d\right) = u P_d$

Rearrange (elim P_d, v) to get $\dfrac{\rho_d}{\rho_u} = 4$ (or $\dfrac{\rho_d}{\rho_u} = 1$)

Ideal gas law ⇒ $kT_d = P_d \dfrac{\mu m}{\rho_d}$

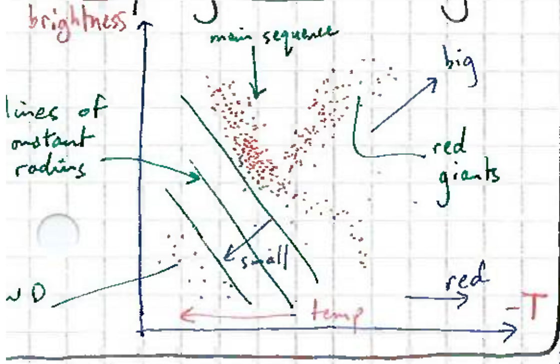
$\therefore T_d = \dfrac{3}{16}\dfrac{\mu m V^2}{k}$

So $T > 10^7\,\mathrm{K}$ for first 1000 yrs - remnants emit X-rays. Get ~ free exp. until swept up mass is ~ ejecta mass.

Then get adiabatic exp.

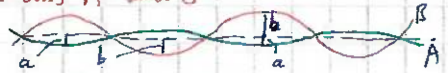
Then as it slows down, post shock gas cools radiatively - merges with interst. medium ~30 pc...

Hertzsprung - Russell Diagram



Discovery of White Dwarfs

Sirius A and B:



b is observed to be ≃ 2a.

Also, know $M_A \simeq 2M_\odot \Rightarrow M_B \simeq M_\odot$

now both stars look white ∴ both have same temp ($\sim 10^4\,\mathrm{K}$).

Luminosities are observed to be $L_A \simeq 10000\,L_B$

know $L = 4\pi R^2 \sigma T^4$

$\Rightarrow R_B \simeq \dfrac{R_A}{100}$

Also ⇒ $\rho_B \sim 10^6 \rho_A$ say $10^9\,\mathrm{kg\,m^{-3}}$

These facts $(T, R, \rho) \Longrightarrow$ Sirius B is a White Dwarf.

Pulsars

Neutron Stars cool by ν emission

$T_c: 10^{11} \rightarrow 10^8\,\mathrm{K}$, $T_{surf} \sim 10^6\,\mathrm{K}$

Emission is in EUV band ∴ is absorbed by H in galaxy....

⇒ don't see neutron stars.

Pulsars were seen 1967 - Flashes of radio waves with periods $1.6\,\mathrm{ms} \lesssim P \lesssim 5\,\mathrm{s}$ and $\dot{P} > 0$

also, radius is small

So $\Omega^2 \lesssim \dfrac{GM}{R^3} \sim G\rho$ (ie $I\Omega^2 = GPE$)

ie ρ is high ⇒ N.S.

Model as rotating magnetic dipole:

Power $= \dfrac{\mu_0 |\ddot{m}|^2}{6\pi c^3}$

where $m = \dfrac{B_{poles} R^3}{2}\left(\dfrac{4\pi}{\mu_0}\right)\sin(\Omega t)$

to get m_{eff}, project onto our dir'n:

$m_{eff} = m\sin\alpha$ and $|\ddot{m}| = m\sin\alpha\,\Omega^2$

So $\boxed{P = \left(\dfrac{4\pi}{\mu_0}\right)\dfrac{2B_p^2 R^6 \Omega^4 \sin^2\alpha}{3c^3}}$

now $P = -\dfrac{dE_{rot}}{dt} = -\dfrac{d}{dt}\left(\dfrac{1}{2}I\Omega^2\right)$!

So $\boxed{P = -\dot{E}_{rot} = I\Omega\dot{\Omega}}$

Can estimate age ie "spin down time"

as $\boxed{\dot{\Omega} \sim -\dfrac{\Omega}{\tau} \Rightarrow \tau = \dfrac{6Ic^3/\mu_0}{B_p^2 R^6 \sin^2\alpha\,\Omega_0^2 . 4\pi}}$

Can construct "braking index", n, s.t. $\dot{\Omega} \propto -\Omega^n$ and $\boxed{n = -\dfrac{\ddot{\Omega}\Omega}{\dot{\Omega}^2}}$

For magnetic dipole, n = 3. For crab pulsar, n = 2.5.

CRAB PULSAR: nebula found to emit X-rays - needs injection of $P \sim 5 \times 10^{31}\,\mathrm{W}$. If use P, $\dot{P}$ data, assume M, R (ie I) then get $P \sim 6 \times 10^{31}\,\mathrm{W}$! (also get B_p - not too strong)

Most of power emitted is Low freq (Ω) waves - excites ionised interstellar medium eg causes blue light in Crab nebula. Small fraction is the pulses (radio). Sometimes get X-ray or γ-ray pulses. Pulse emission: B is moving ∴ get $\underline{E} = \underline{v} \times \underline{B}$ ie voltage gap which rips charges of surface - acc'd ∴ emit

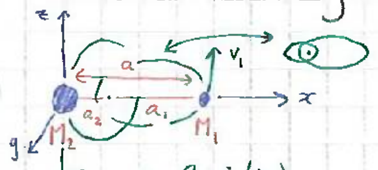
As P increases field decreases - eventually, pulsar dies.

Glitches can occur - probably due to crust quakes. Takes some days for fluid interior to equilibrate.

$\dfrac{\Delta P}{P}$ can be $\sim 10^{-8}$ or $\sim 10^{6}$...

Few pulsars actually known to be associated with SN remnants - Type I or BH or beam misses earth etc.

Mass Transfer in Binary Systems

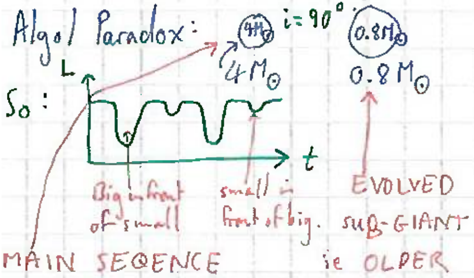


$V_2 = a_2 \Omega \sin(inc)$

Mass f'n $f_1 = \frac{(M_2 \sin i)^3}{M^2} = \frac{V_1^3}{\Omega G}$ from Kepler

So $\frac{f_1}{f_2} = \left(\frac{M_2}{M_1}\right)^3 = \left(\frac{V_1}{V_2}\right)^3$ $[M_2 V_2 = M_1 V_1]$

Algol Paradox:

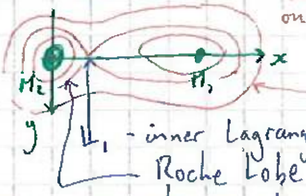


But more massive stars evolve quicker! ⇒ ∃ mass transfer

Grav pot + centrifugal, $\phi = \frac{-GM_2}{(x^2+y^2+z^2)^{1/2}} - \frac{GM_1}{((x-a)^2+y^2+z^2)^{1/2}} - \frac{\Omega^2}{2}\left((x-\mu a)^2+y^2\right)$ $\mu = \frac{M_2}{M_1+M_2}$

distance out from c.o. mass.

If plot this, it looks like:



L_1 - inner Lagrangian point. Roche Lobe overflow is one way - also, one star can have a strong wind. Total angular mom:

$$J = M_1 a_1^2 \Omega + M_2 a_2^2 \Omega$$
$$= \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega$$

First approx: assume conservative ie $\dot{J} = 0$ and $\dot{M}_{tot} = 0$.

Kepler ⇒ $\frac{\dot{a}}{a} = -\frac{2\dot{\Omega}}{3\Omega}$ subst into $\dot{J} = 0$

$$\Rightarrow \boxed{-\frac{\dot{\Omega}}{\Omega} = \frac{\dot{P}}{P} = \frac{3\dot{M}_1 (M_1 - M_2)}{M_1 M_2}}$$

If M_1 loses mass ($\dot{M}_1 < 0$) and $M_1 < M_2$ as in dig, $\dot{a} > 0$ ∴ less mass transfer occurs - stable. But if $M_1 > M_2$ and M_1 loses mass $\dot{a}$ is -ve and more mass transfer occurs - usually stabilises when $M_1 \simeq M_2$.

If one star is WD and P = few hrs, get flare up by ×100 every few weeks as waves of matter are accreted known as a nova

[Sometime accreted H fusions and see bright burst. If gains more than loses, get Type I Supernova

If have main seq. star + neutron star or BH then have X-ray binary.

$$L_{grav} = \frac{GM\dot{M}}{R} = 4\pi R^2 \sigma T_{surface}^4.$$

Get roughly 2 types: Low Mass XB companion is $M < 1 M_\odot$

High Mass XB: companion is $M \gtrsim 10 M_\odot$

For XB to form need supernova to occur in less massive star otherwise binary system becomes unbound: get runaway star + high speed X-ray binary.

Accretion

Matter must be bound to the star ⇒ $\frac{1}{2}mv^2 \leq \frac{GMm}{r}$ can express in terms of an accreation radius $\boxed{R_A = \frac{2GM}{v^2}}$

Free fall density:

$\rho_{ff} \simeq \frac{\dot{M}}{4\pi r^2 V_{ff}}$ where $V_{ff} = \sqrt{\frac{2GM}{r}}$ so $\rho_{ff} \propto r^{-3/2}$

Mass ⟶ energy conversion efficiency: ε

$\boxed{\varepsilon = \frac{R_S}{2R}}$ WD : $\sim 10^{-4}$ NS : ~ 0.15

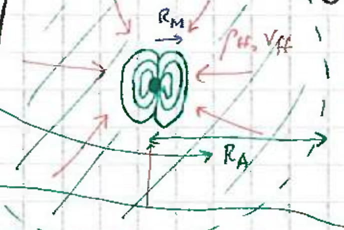
Minimum temp. of accreting gas is black body temp: $T_{BB} = \left(\frac{GM\dot{M}}{4\pi R^3 \sigma}\right)^{1/4}$ or $L \gtrsim 1000 L_\odot$

get WD 2×10^5K NS 6×10^6K.

Maximum temp of accreting gas is shock temp.: $T_s = \frac{3}{16}\frac{GMm_p}{kR}$ WD 2×10^8K NS 2×10^{11}K

If lots of matter present, can interact with photons and equilibrate - so ⟶ black body spectrum.

Accretion onto Magnetised N.S. or W.D.

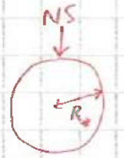


Assume sph. sym. flow, dipole field.

Mag. field dominates flow in reigion R_M where mag. pressure/stress > pressure of flow.

$P_{field} = \frac{B^2}{2\mu_0} = \frac{m^2}{32\pi^2 R_M^6}$ for dipole

$P_{flow} = \rho V_{ff}^2 = \frac{\dot{M}}{4\pi^2 R_M^2}\sqrt{\frac{2GM}{R_M}}$



$P_{field} = P_{flow} \Rightarrow R_M \propto L^{-2/7} m^{4/7}$ using $L = \frac{GM\dot{M}}{R_*}$

If flow is not sph. sym and disc like, get $R_M^{disk} \simeq \frac{1}{2} R_M^{sph.}$

Objects accrete ang. mom - they spin up until co-rotating with inner edge of disc. $\dot{P} < 0$!

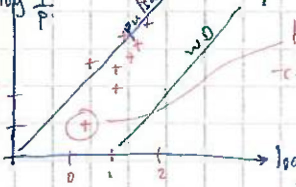
Can get pulses in HMXB's: N.S. accretes matter onto poles coz mag. field funnels it down. Poles get hot + become a torch.

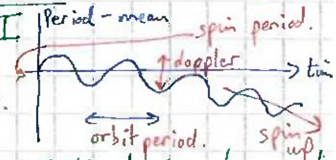
Now $\dot{J} = \cancel{\dot{I}\Omega} + I\dot{\Omega} = \underbrace{\dot{M}\sqrt{GMR_M}}_{\text{from Kepler}} \Rightarrow \dot{P} = -\frac{P^2 \dot{M}\sqrt{GMR_M}}{2\pi I}$

small

But can use this to get $\frac{\dot{P}}{P} \propto -\frac{PL^{6/7} m^{2/7}}{I}$

So plot:

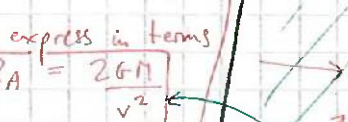




Most LMXB donot pulse - weak mag field maybe. But can get "Quasi-Periodic Oscillations" in X ray flux. Maybe coz beating between spin freq. of NS and orbit freq. of acc. disc

X-ray bursts

X ray flux:



due to fusion burning of accreted matter: He burning.

expect $\frac{\text{usual accretion flux}}{\text{time av. burst flux}} = \frac{200 \text{ MeV}}{6 \text{ MeV}} \sim 30$ - see 50→200

If know dist can use $L = 4\pi R^2 \sigma T^4$ to get temp and radius.

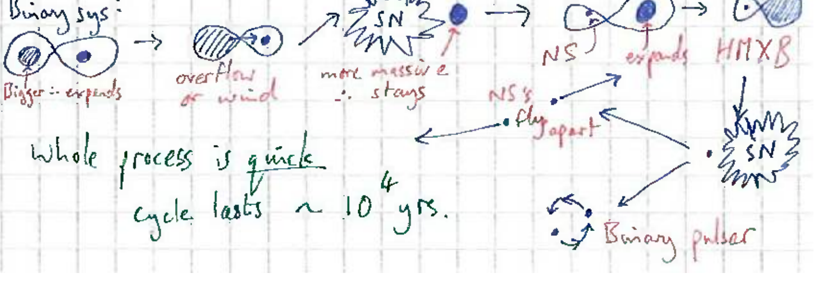
Eddington Limit

ie When radiation pressure balances pressure due to gravity. Any object lasting > a few dynamical times (ie not supernovae) should have $L < L_{edd}$. $F_{grav} = \frac{GMm_p}{R^2}$ (acting on proton)

$F_{rad} = \underbrace{\frac{L}{4\pi R^2 h\nu}}_{\text{flux of photons of mean energy } h\nu} \sigma_T \cdot \underbrace{\frac{h\nu}{c}}_{\text{momentum of photon.}}$ (acting on an electron) $\Rightarrow \boxed{L_{edd} = \frac{4\pi GMm_p c}{\sigma_T}}$

$= \sim 10^{31}$ W per $M_\odot$

Evolution of HMXB's



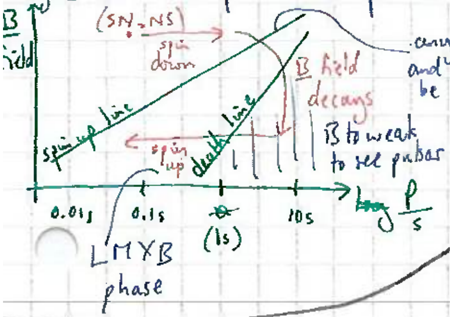
Whole process is quick cycle lasts ~ 10^4 yrs.

G. A. and C. ⑥

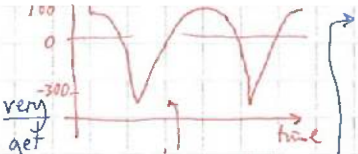
Evolution of LMXB's

10 stars with a few pc! Milky way has >100 orbiting it.

This less clear than HMXB.
Many exist in globular clusters
High star density ∴ dead NS. can tidally capture main sequence star and spin up to ~ ms. Young NS no can do as $\underline{B}$ is strong ∴ R_M large ∴ corotates very soon. Old NS has $R_M \simeq R_*$ coz $\underline{B}$ weak.
⇒ Accretion is v. long lived.
After spin up accretion decreases - maybe even companion evaporates.



Binary Pulsars



Can get mass f'n very accurately. Also can get doppler + grav redshift to relate M_1, M_2, a, e. Also, precession of periastron, ω, is $4.2^\circ yr^{-1}$ $\dot{\omega} = \frac{6\pi G M_2}{a_1(1-e^2)P_c^2}$

⇒ have closed solution to prob. - Tests GR.
$\dot{a} < 0$ - orbit tightens at rate consistent with grav waves.
Dipole emission forbidden by cons. of angular momentum
Quadrupole emission:

$$\boxed{L_{GW} = \frac{G}{5c^5}\langle \dddot{I}^2 \rangle}$$

$\dot{I}$ - rate of change of moment of inertia.

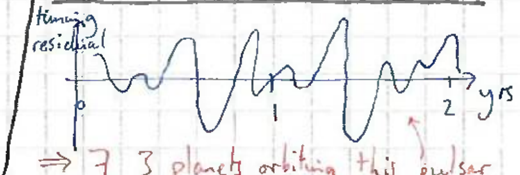
$$\left.\begin{aligned}\dddot{I} &\sim \frac{MR^2}{T^3}\\ &\sim \frac{Mv^3}{R}\end{aligned}\right\}\text{dimensional arg.}$$

$\Rightarrow L_{GW} \sim \frac{c^5}{G}\left(\frac{R_S}{R}\right)^2\left(\frac{v}{c}\right)^6$
very big
So big effect for compact objects moving fast
$\underline{\underline{SN}} \rightarrow \frac{\Delta l}{l} \sim 10^{-18}$ on earth.

$E_{orb} \sim -\frac{GM^2}{2a}$

$\underline{\text{Now}}$: Kepler: $\frac{\dot{P}}{P} = \frac{3}{2}\frac{\dot{a}}{a} = -\frac{3}{2}\frac{E}{\dot{E}}$ ...
$\frac{GMm}{R^2} = MmR\Omega^2$ subst for T
get $L_{GW} = \frac{G^4M^5}{c^5R^5} = \dot{E}$
⇒ find estimate for $\dot{P}$ fits with G.R.

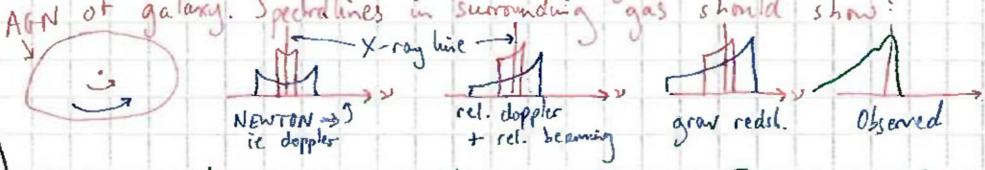
Planets around Pulsars



⇒ ∃ 3 planets orbiting this pulsar
detect doppler shift due to walking speeds

Active Galactic Nuclei

AGN is small centre of galaxy that outshines rest of gal ($\sim 10^{11}$ stars)
$L = 10^{36} - 10^{40}$ W. (ie $\sim 10^{13} L_\odot$. Milky Way is $10^{11} L_\odot$)
Spectra ⇒ not ordinary stars. Variation timescale ⇒ small
High L AGN's: Quasars Low L AGN's eg Seyfert Galaxies.
V. important for cosmology - very high redshift. (~5)
Very broad, strong spectral lines ⇒ v. high gas speeds close to AGN: the broad-line region.
If all grav. energy released, $L = \frac{GM\dot{M}}{R}$ but if some sucked in, $L = \varepsilon \dot{M} c^2$ typically $\varepsilon \ll 0.1$. (0.1 = "standard value")
Eddington limit ⇒ constraint on the mass of AGN.
SPECTRA: Surface area of disk $\propto M^2$ ⇒ emission per $m^2 \propto \frac{\dot{M}}{M^2} \propto \frac{1}{M}$
but emission per $m^2 = \sigma T^4$ so temp $\propto \frac{1}{M^{1/4}}$
So for high mass ⇒ primary emission is UV which can be absorbed by gas clouds and emitted as visible - fits with observations.
∃ AGN flares like solar flares due to reconnection of $\underline{B}$, acc'n of particles....
FURTHER EVIDENCE FOR BH: if occur anywhere, will be at centre of galaxy. Spectral lines in surrounding gas should show:

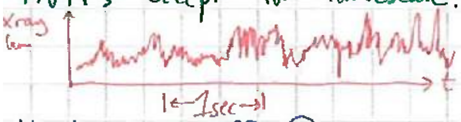


In some galaxies - have actually seen orbits at $10^5 R_s$, stars with high speeds ... but low luminosity ∴ central mass prob. not stellar material eg in Milky Way

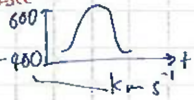
$f = 0.25 M_\odot$
$M \gtrsim 6 M_\odot$
$K_1 = 70\ kms^{-1}$

Accreting (Stellar Mass) Black Holes

eg Cygnus X-1 is prob. a black hole because: high mass (from spec. evidence) small size (timescales of variation ~ms)
-shows chaotic flickering hard X-ray spectrum unlike NS X-ray binaries by very like AGN's except for timescale.



Also large vel: ⇒ high mass companion.



Quasar Lifetimes

Time taken to double mass if radiating at Ed. limit:
$\frac{M}{\dot{M}} = \frac{M}{L_{Ed}/\varepsilon c^2} = \frac{c\sigma_T\varepsilon}{4\pi G m_p} = 4\times10^8\varepsilon$ yrs.
Characteristic lifetime for Quasars is $\sim\frac{1}{100}$ of Hubble time and they are seen in ~1/100 galaxies so maybe every galaxy has one!

Jets and Radio Sources

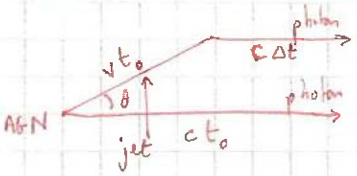
In some AGN's, mag. field + electrons are squirted into jets which emit radio waves, (synchrotron radiation)
We know V - volume of jet source and P - total power. One e^- emits synchrotron radiation at power $p \propto \gamma^2 B^2$ at frequency $\gamma^2 B$

Magnetic energy is B^2V

page 1 of last small writing lecture.

Superluminal Motions and Doppler Beaming.

-evidence that jets flow out at close to c.



time difference when detect blob of jet at diff. pos'ns in sky is Δt.

So $v_{apparent} = \frac{vt_0 \sin\theta}{\Delta t}$ but $ct_0 = vt_0\cos\theta + c\Delta t$

$$\Rightarrow \boxed{v_{app} = \frac{v\sin\theta}{1-\frac{v}{c}\cos\theta}}$$

maximum v_{app} at $\sin\theta = \frac{1}{\gamma}$ so big effect for small θ ...

Intensity is increased (aswell as blueshift). For Power $\propto \frac{1}{\nu^\alpha}$,

$$P_{observed} \propto \frac{1}{\left[\gamma(1-\frac{v}{c}\cos\theta)\right]^{3+\alpha}}$$

So substantial fraction of radio jets display superluminal motions.

Gamma-Ray Bursts.

-Bursts of γ-rays (photons of ~1 MeV) lasting about 0.1 - 30 seconds. Flux high ~ same as Optical flux from Venus at its brightest. After glows lasting ~ months (optical + radio) show spectral lines with moderate redshift $\Rightarrow$ bursts are "cosmological". Rate is: ~ 1 perday (10^{-6} per yr per galaxy) Energy release is ~ 10^{45} J in a few seconds. Amount is no prob but time is.

NS + BH collisions in binary systems are poss. Then: energy released as neutrinos in core can collide in outer regions to give e^+e^- fireball. or magnetic fields (need strong 10^{11}T) channel rotational energy in to a strong outflow. ∃ evidence ⇒ emitting region is relativistically expanding - see notes.

Distance Definitions.

PROPER DISTANCE: $d_{prop} = \int ds$ at const. t, θ, ϕ

ie $\boxed{d_{prop} = R(t_0)\chi}$ ie lay ruler down at time t = now.

χ coord of comoving galaxy.

RADAR DISTANCE: $\boxed{d_{radar}(t_0) = \frac{1}{2}c(t_2 - t_1)}$ $t_0 = \frac{t_3+t_1}{2}$

ie send out light pulse, reflect off galaxy.

LUMINOSITY DISTANCE: In Euclidean universe, can say $d_L = \left(\frac{L}{4\pi F_{euc.}}\right)^{1/2}$. But F is changed in 3 ways:

① $4\pi r_{euc}^2 \neq 4\pi r_{prop}^2$. Proper area $= 4\pi(\sigma R(t_0))^2$ $(r = \sigma R)$
$\therefore = 4\pi[R_0 S(\chi_1)]^2$ — χ coord of comoving galaxy.

2) Photons are redshifted: $\nu_{obs} = \nu_{em} \cdot \frac{R(t_1)}{R(t_0)} = \frac{\nu_{em}}{1+z}$
Luminosity $\propto \nu$ so L is reduced by $\frac{1}{1+z}$.

3) Rate of arrival is reduced - same redshift arg. $\propto \frac{1}{1+z}$

$$\Rightarrow F(t_0) = \frac{L(t_1)}{4\pi(R_0 S(\chi_1))^2}\frac{1}{(1+z)^2}$$

$$\text{now } \boxed{d_L(t_0,\chi_1) = \left(\frac{L(t_1)}{4\pi F}\right)^{1/2} = R_0 S(\chi_1)(1+z)}$$

- note - dep on history of universe through χ because $\chi = \int \frac{c\,dt}{R(t)}$.

Transforming FRW Metric

FRW metric is: $ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

We look out radially so let's put nasty bit into angular part: let $d\chi = \frac{d\sigma}{\sqrt{1-k\sigma^2}}$ then:

$$\boxed{ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]}$$

where $\chi = \sin^{-1}\sigma$ for $k = +1$ and $S(\chi) = \sin\chi$ $k = +1$
$= \sigma$ for $k = 0$ $= \chi$ $k = 0$
$= \sinh^{-1}\sigma$ for $k = -1$ $= \sinh\chi$ $k = -1$.

Redshift from FRW metric

Radial photon propagation gives us $\frac{d\chi}{dt}$ as $ds^2 = 0$, θ, ϕ const

ie $dt^2 - \frac{R^2}{c^2}d\chi^2 = 0$ $\Rightarrow \boxed{\frac{d\chi}{dt} = -\frac{c}{R}}$

So $\underbrace{\chi(t_0)}_{\text{let this be origin ie } = 0} - \chi(t_1) = \int_{t_1}^{t_0} -\frac{c\,dt}{R}$

Now $\chi_1 = \int_{t_1}^{t_0} \frac{c\,dt}{R} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{c\,dt}{R} = \int_{t_1+\delta t_1}^{t_1} + \cancel{\int_{t_1}^{t_0}} + \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{R}$

time-translation invariance. same but not cancelled!

$$\Rightarrow \int_{t_1}^{t_1+\delta t_1} \frac{c\,dt}{R} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{R} \Rightarrow \boxed{\frac{\delta t_1}{\delta t_0} = \frac{R(t_1)}{R(t_0)}}$$

So redshift, $\boxed{z + 1 = \frac{R_0}{R_1}}$ ie $\frac{\text{scale factor when receive}}{\text{scale factor when emit.}}$

ANGULAR DIAMETER DISTANCE In E^3, $\Delta\theta = \frac{D}{d}$

now, $d_\theta = \frac{\text{assumed proper size, } D}{\text{measured angular diam, } \Delta\theta}$

ie:

From angular part of metric, $D = R_1 S(\chi_1)\Delta\theta$

So $d_\theta(t_0, \chi_1) = R_1 S(\chi_1) = \frac{R_0 S(\chi_1)}{1+z}$ as $\frac{R_1}{R_0} = \frac{1}{1+z}$

$$\Rightarrow \boxed{d_\theta = \frac{R_0 S(\chi)}{(1+z)}}$$

Overview of field eq'ns

Hom + Isot. + expansion ⇒ FRW

Only free params are R(t) and k. Now relate to gravitating matter. Curvature of space-time is crucial.

The Cosmological Field Equations

Take FRW metric and let $\theta, \phi = \text{const}$. Then have 2-D surface - use Gauss's theorem to find curvature: gives:

$$\boxed{K(t) = -\frac{\ddot{R}}{R}}$$

We use metric where interval has units $t^2 \therefore [K] = \frac{1}{s^2}$

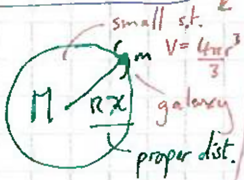
$\boxed{\text{Assume } K \propto \rho}$

Then use dim. anal: $K(t) = C\rho(t)G^x c^y$

$\underline{x = 1}$ $\underline{y = 0}$

What is C ?

use Newtonian limit: $m\ddot{R}\chi = -\frac{GMm}{(\chi R)^2}$ for



but $M = \frac{4\pi}{3}R^3\chi^3\rho$

$\Longrightarrow \frac{\ddot{R}}{R} = -\underbrace{\frac{4\pi}{3}}_{C}G\rho$ But we have neglected pressure

can estimate contribution using dimensions → ratio of strengths $= \frac{p}{\rho c^2} << 1$ (non rel)

But for relativistic systems, eg photons $p = \frac{1}{3}U$ ← energy density ⇒ 3 contribution.

Relation between ρ and p is equation of state:

$$\boxed{\varepsilon \cdot \frac{\rho c^2}{3} = p} \quad \text{ie } \varepsilon = \frac{3p}{\rho c^2}$$

So $\varepsilon = 1$ if universe full of radiation, $= 0$ if matter.

So if use this, get: $\boxed{\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\varepsilon) = 0}$

ALSO have assumed here that if $\rho = 0$, $K = 0$! ie flat Minkowski space time. If not, let $K \propto \rho + \text{const}$

$$\Longrightarrow \boxed{\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\varepsilon) - \frac{\Lambda}{3} = 0}$$ Force eq'n.

Static Universe

Assume dust filled, $\varepsilon = 0$

$\ddot{R} = \dot{R} = 0$ then $\boxed{\Lambda = 4\pi G\rho}$

and $-\frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}$

$\Rightarrow R = \sqrt{\frac{kc^2}{4\pi G\rho}}$ only real solutions for $k = 1$ ∴ not flat ....

experimentally $\rho(t_0) \simeq 3 \times 10^{-28}\ \text{kg m}^{-3}$

$\Rightarrow R \simeq 6 \times 10^{26}\ \text{m} = 20\,000\ \text{Mpc}$ ∴ possible ie > known universe. Also $\rho_0 \Rightarrow \Lambda = 2.5 \times 10^{-37}\ \text{s}^{-2}$ which evades current measurement (10^{-33})

But: (1) unstable - small pert. lead to runaway expansion or collapse (2) the universe is observed to be expanding

So set $\Lambda = 0$

Define DECELERATION PARAMETER: $q(t) = -\frac{\ddot{R}R}{\dot{R}^2}$

ie let $\dot{R} \propto R^q$ $[q_0 = 0.5 \Leftrightarrow \text{EdS}]$

Can write $F = \frac{LH_0^2}{4\pi(cz)^2}(1 + (q_0 - 1)z + \dots)$

then: use standard candles to plot F vs z for different q_0. But high z galaxies are younger ∴ more luminous ∴ no longer standard candles...

Now, $H^2 = \frac{8\pi G\rho}{3}$

so if have H, → ρ the numbers work out OK. The Einstein-de-Sitter density is called the Critical density now

ENERGY CONSERVATION:

If $\varepsilon = 0$, ie matter dominated, $M = \frac{4\pi}{3}\chi^3R^3\rho$

$\Rightarrow \underline{\underline{\rho \propto \frac{1}{R^3}}}$ ie consider sphere expanding - χ const, M const.

If $\varepsilon = 1$ must take account work done during expansion:

$$\rho_{new}c^2 V_{new} = \rho_{old}c^2 V_{old} - pdV$$

ie $(\rho + d\rho)c^2(V + dV) = \rho c^2 V - pdV$

But $V = \frac{4\pi}{3}\chi^3R^3$ and $p = \frac{\rho c^2}{3}$

$\Rightarrow \underline{\underline{\rho \propto R^{-4}}}$ Now, $\rho = \frac{N}{V}h\nu = nh\nu \propto \frac{1}{R^4}$

but $n \propto \frac{1}{R^3} \Rightarrow \boxed{\nu \propto \frac{1}{R}}$ Redshift law!

Also, $\rho = aT^4 \Rightarrow \boxed{T \propto \frac{1}{R}}$ early universe had hot CMBR!

Generally, $\rho \propto \frac{1}{R^{(3+\varepsilon)}}$ but now need GR to get:

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}}$$ Energy eq'n

$k = -1, 0$ or 1

Einstein-de-Sitter Universe + Friedmann models

EdS: Spatially flat ie let $k = 0$ then:

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3}}$$

using this $\Rightarrow \dot{R}^2 \propto R^{2-(3+\varepsilon)}$

ie $R \propto t^{\frac{2}{3+\varepsilon}}$

So for matter dominated un. $R \propto t^{2/3}$

for radiation dominated un. $R \propto t^{1/2}$

Current universe is closest to $\Lambda = \varepsilon = k = 0$, the Einstein-de-Sitter universe - Euclidean spatial hypersurface

Define HUBBLE PARAMATER $\boxed{H(t) = \frac{\dot{R}}{R}}$. now $d_{proper} = R\chi$

and for comov. $\dot{d} = \dot{R}\chi$

So $\dot{d} = H\underbrace{R\chi}_{d}$ ie the regular Hubble Law

Now $\chi = \int_t^{t_0}\frac{c\,dt}{R(t)}$ but $\frac{dt}{R} = -\frac{dz}{R_0}\cdot\frac{R}{\dot{R}} = -\frac{dz}{R_0 H}$

So $\chi = \int_0^z \frac{c\,dz}{R_0 H(z)}$ ← have changed $\int$ over time to $\int$ over redshift.

Define: DENSITY PARAMETER $\Omega = \frac{\text{actual}}{\text{critical}}$ density

ie $\Omega = \frac{8\pi G\rho(t)}{3H^2(t)}$ then $\boxed{q = \frac{\Omega}{2}(1+\varepsilon)}$ so now, $q_0 = \frac{\Omega}{2}$

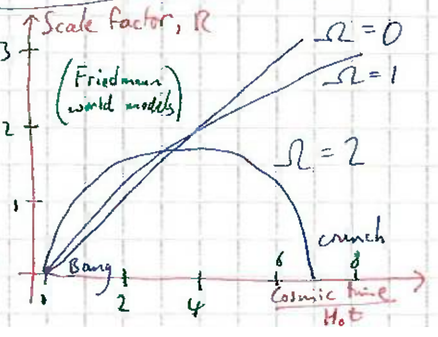
also, $\boxed{\Omega - 1 = \frac{kc^2}{H^2R^2}}$

So $\Omega > 1 \Rightarrow k$ +ve

$\Omega = 1 \Rightarrow k = 0$

$\Omega < 1 \Rightarrow k$ −ve

We have done everything for $k = 0$. Can do properly and find that Ω controls the dynamics too.



The C.M.B.

Know BBR (coz in T.E.! with matter) when emitted but why is it still BBR now?

Consider comoving volume ie proper volume $\propto R^3$. Let t_1 be just after recomb.

now: number in $V\big|_{t_1}$ = number in $V\big|_{t_0}$

ie $n_{\nu_1}(t_1)\,d\nu_1(t_1)\,V(t_1) = n_{\nu_0}(t_0)\,d\nu_0(t_0)\,V(t_0)$

now redshift ⇒ $\nu \propto \frac{1}{R}$ and $d\nu \propto \frac{1}{R}$

So

$$\underbrace{n_{\nu_0}(t_0) = n_{\nu_1}(t_1)}\cdot\left(\frac{R_1}{R_0}\right)^2$$

So - has same form - diff amount....

but $\nu_1 = \frac{R_0}{R_1}\nu_0$ so in exponent, $\frac{h\nu_1}{kT_1}$ becomes $\frac{h\nu_0}{kT_1\frac{R_1}{R_0}}$ $\quad T_0 = \frac{T_1R_1}{R_0} = \frac{T_1}{1+z}$

Photon to Baryon Ratio:

Total no. of photons (per m^{-3}) is

$$N = \int_0^\infty n_\nu\,d\nu = \frac{4.2\times10^8\,m^{-3}}{}$$

for $T_0 = 2.76$ K

But matter density is ~ 0.7 protons m^{-3}

$$\Rightarrow \frac{N\gamma}{N_{bar}} \sim 10^9$$

BBR: $n_\nu = \frac{8\pi\nu^2}{c^3(e^{h\nu/kT}-1)}$ energy density per unit freq, u_ν

and intensity $I_\nu = \frac{c\,n_\nu h\nu}{4\pi}$

Now: to find temp of recombination, must use Saha eq'n which accounts for fact that recomb. no take place in an instant...

can estimate via saying there must be less than one photon (of energy 13.6 eV) per baryon:

$$e^{\frac{-13.6eV}{kT_r}} \sim 10^{-9}$$

$\Rightarrow T_{rec} = 7600$ K

better answer is 4000 K.

Angular Size of Imprints

Def'n of angular diamter distance:

$d_\theta = \frac{D}{\Delta\theta}$ — proper $\therefore \Delta\theta = \frac{D(1+z)}{R_0 S(\chi)}$

For a dust filled universe,

$S \simeq \frac{2c}{R_0H_0\Omega}$ (and if $z \gg 1$ $1+z \simeq z$) then $\Delta\theta = \frac{H_0\Omega Dz}{c}$

let's work out angle substended by what is now a cluster of galaxies:

$M = \frac{4\pi}{3}D^3\rho(z) = \frac{4\pi}{3}D^3\rho_0(1+z)^3$ from scale factor

$\Rightarrow \Delta\theta \sim 10$ minutes.

ie progenitor of cluster subtends 10′ at recomb.

Observed fluctuations are ~ 10° ! (later)

Particle Horizons

Def'n:

A particle horizon is the χ coord of the most distant object that can be seen at a given time, χ_p

If $\chi_p < \infty$ ∃ parts of universe not accesible at a given time. χ is $< \infty$ as shown: big bang: $t=0$ just after, $t = t_1$, now $\chi_p = \int_0^{t_1}\frac{c\,dt}{R(t)} \propto \int_0^{t_1}\frac{dt}{t^{\frac{2}{3+\epsilon}}}$ valid ∀ Friedmann models not just k=1

This is finite if $\frac{2}{\epsilon+3} < 1$

In EdS, ∃ always a χ_p.

But Big Bang ⇒ all pts were one, once? if early enough $-\frac{c^2k}{R^2}$ big.: Yes but also curvature = ∞ so ∄ a common Lorentz frame to compare speeds...

Now χ_p at $t_0 = \int_0^{t_0}\frac{c\,dt}{R} = \frac{3ct_0}{R_0}$ for EdS

So $D_{particle}$ (proper dist) = $3ct_0$ For EdS, $H_0 = \frac{2}{3t_0}$

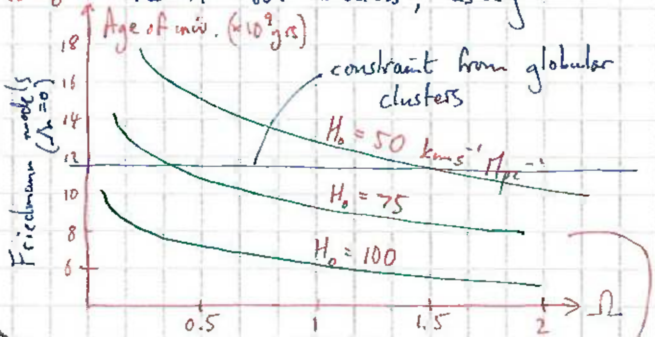
Redshift and χ_p: $\chi(z) = \int_0^z\frac{c\,dz}{R_0\,H(z)}$

but in EdS $H(z) = H_0(1+z)^{3/2}$ so

$$\chi(z) = \frac{2c}{R_0H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

but if $\chi = \chi_p$ (using this) so $\chi_p = \frac{2c}{R_0H_0}$ then ⇒ $\underline{z = \infty}$. i.e. objects on χ_p are at ∞ redshift.

Now, age of universe can restrict our models, using



The Age of the Universe

GLOBULAR CLUSTERS Expect to be v. old as low metal content

~10^5 stars born at same time from same cloud etc... so rel. pos'n on HR diag only dep on M. Turn off point gives lower bound on age of univ.

average over many clusters → 12.5 ± 1.5 × 10^9 yrs.

RADIOACTIVE DATING eg Uranium in SN formed by rapid addition of neutrons gives $\left[\frac{U^{235}}{U^{238}}\right]_{initial}$. We know decay rates ⇒ can get another lower bound.

PHYSICAL METHODS - Bypass distance ladder.

~~CMB~~ Sunyaev-Zel'dovich effect: CMB photons get and increase in mean energy when pass through hot intracluster gas ⇒ T↓ by 1 mK in low freq. region of spe. :

Measure with radio telescope. (X-ray telescope gets gas details coz bremsstral.)

→ 42 ± 10 - low. Uncert: depth /size of cluster....

Grav. lenses - get time delay from Quasar variability. If know mass etc of lens → get H_0 $(t \propto \frac{1}{H_0})$ but knowing mass is hard.

The Hubble Constant

DISTANCE LADDER METHODS Use Cepheid Var's $10^4L_\odot$, massive. Atmospheres not in Hydrostatic eq ⇒ pulsate - Satisfy Period ∝ Luminosity. ⇒ get distance if know const from parallax to Cepheid in own galaxy. Then can use to calibrate say SN then observe SN far away where can't see Cepheids. etc etc... HST helps. ↳ HST can see Cepheids at Cosmologically interesting distances - gives 80 ± 17 km s⁻¹ Mpc⁻¹. These methods give 50 - 100 km s⁻¹ Mpc⁻¹. Uncertainty is in effect of cluster's gravity on light from Cepheids.

TYPE 1A SUPERNOVAE - good standard candles. - currently being calibrated using HST - reach out to distances where Hubble flow ≫ peculiar velocity! Results are less: 57 ± 3 and 65 ± 6 ! good

Thermal History of Universe

$h = \frac{H_0}{50\,\text{kms}^{-1}\text{Mpc}^{-1}} = \frac{H_0}{1.62 \times 10^{-18}\,\text{s}^{-1}}$

Know : radiation : $T \propto \frac{1}{R}$ $\left(\text{gas } T \propto \frac{1}{R^2}\right)$

Know ρ_{m0} $\rightarrow = \underline{4.7 \times 10^{-27} \times \Omega h^2}$ (S.I)

current matter density

$\rho_{rad\,0}$ comes from CMB predominantly - equiv. mass-energy density is $\sim \frac{\rho_{m0}}{1000}$ (!)

But ρ_{rad} scales $\propto T^4$

so for $R\downarrow$, $T\uparrow$. ⇒ early times, important.

Important number : $\frac{n_{baryon}}{n_{photon}} = 7 \times 10^{-9}\,\Omega_m h^2$

now $\rho_{total} = \rho_{m0}\left(\frac{R_0}{R}\right)^{-} + \rho_{rad\,0}\left(\frac{R_0}{R}\right)^{'}$

So the ephoch of equality occurs when scale fact is such that $\rho_{m0}\left(\frac{R_0}{R}\right)^3 = \rho_{rad0}\left(\frac{R_0}{R}\right)^4$

$\Rightarrow R_{eq} = R_0 \frac{\rho_{rad_0}}{\rho_{m0}}$ ⇒ redshift $z_{eq} \sim 10^5 \Omega_m h^2$

So before, radiation dominant, after, matter dominant

Helium and Deuterium Synthesis

These 3 processes convert $n \rightleftharpoons p$:

β decay : $n \longrightarrow p + e^- + \bar{\nu}_e$

inverse β decay : $p + e^- \rightarrow n + \nu_e$

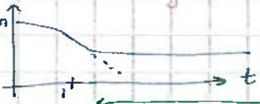
and : $p + \bar{\nu}_e \rightarrow n + e^+$

$\underbrace{\qquad\qquad\qquad}_{\text{small cross-sections}}$

Thermal eq'n ⇒ $\frac{N_n}{N_p} = e^{\frac{-(m_n - m_p)c^2}{kT}}$ $\left(\simeq e^{-\frac{1.5}{T_{10}}}\right)$ ⇒ $\frac{\dot{R}}{R} = -\frac{\dot{T}}{T}$

⇒ time taken to establish $\sim 2.5\,T_{10}^{-5}$ secs.

But universe expands $\sim T_{10}^{-2}$ secs. So expansion time scale is smaller than T.E. scale ie density decreases too fast - reachin → T.E. can't happen... $\frac{N_n}{N_p}$ freezes out :



Deuterium formation : $n + p \rightleftharpoons D + \gamma$ But as photons outnumber Baryons must wait until few photons above the dissociation level (2.2MeV) ie. $T \sim 10^9$ K.

When this happens, can get He formation :

$D + D \longrightarrow {}^3He + n\ (\longrightarrow {}^3H + p)$

$D + D \longrightarrow {}^3H + p$ straight away

and ${}^3H + D \rightarrow {}^4He + n$

Provided $\Omega \gtrsim 10^{-3}$, all $D \rightarrow He$. Resultant He mass frac. is $Y = \left(\frac{2N_n}{N_n + N_p}\right)$

Y is dep on :

(i) $\frac{n_b}{n_\gamma}$ ie density of universe through (ii) No. of ν species (iii) neutron ½ life $\tau_{1/2}$

Solution of radiation dominated field equations

Early universe : can drop curvature term (using Retarange(B, ie. $\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2$ but $\rho \propto R^{-(3)+\epsilon)}$

So have $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3}$. If rad. dom, $\rho = \frac{aT^4}{c^2}$ and $T \propto \frac{1}{R}$

$\Rightarrow T = \left(\frac{3c^2}{32\pi Ga}\right)^{1/4} t^{-1/2}$ $\left(\Rightarrow t = 2.3\left(\frac{10^{10}}{T}\right)^2 s\right)$

Let $E = k_B T$ then can c.f. GUT energy scale 10^{14} GeV

→ gives $t_{GUT} = 2 \times 10^{-34}$ seconds.

$t_{e^+e^-\text{ pair creation}} = 1$ sec.

High energy ⇒ neglect mass then $\rho c^2 = \frac{g}{(2\pi\hbar)^3}\frac{4\pi}{c^3}\int_0^\infty \frac{E^3\,dE}{e^{E/kT} \pm 1}$

no. of spin states

fermions or bosons

So Photons : $\rho c^2 = aT^4$ $\left(a = \frac{4\sigma}{c} = \frac{\pi^2 k^4}{15c^3\hbar^3}\right)$

e^- $\frac{7}{8}aT^4$ e^+ $\frac{7}{8}aT^4$

etc. (depending on g)

Effect on density of D / He synthesis

D is destroyed so amount measured now is ≤ original amount.

⇒ $\Omega_{baryons} \lesssim 0.08$. But most of D goes → He so if measure amount of He, get total ... ⇒ $\Omega \gtrsim 0.04$

So $0.04 \lesssim \Omega_b \lesssim 0.08$

So baryons cannot provide critical density !

and " " " dyn. inferred dark matter!

Also, standard cosmological model ⇒ number of ν species is ≤ 4 otherwise He abundance too high.

Also in primordeal nucleosynthesis only have time for 2-body collisions ⇒ only trace amounts of heavy nuclei produced...!

The CMBR again

χ_p ⇒ CMBR regions seperated by >1° were never in causal contact. But we observe $\frac{\Delta T}{T} \sim 10^{-5}$ on 10° scale

Inflation can get around this problem. It predicts a scale invariant power spectrum. That is eg. 10° fluctuations create much larger scale structure than we see but OK coz hasn't had time to form yet.

If universe made only from ordinary matter then amplitude of fluc. too small but if have dark matter, it can form pot. wells then ordin. matter falls in later! Measuring the Power Spectrum of CMB is the key!

Galaxies - Overview

orbits more random in ellipticals

nearest large neighbour is M31 - spiral.

largest ones are elliptical.

radii ~ 1 - 10 kpc

masses ~ $10^8 - 10^{12}\,M_\odot$

luminos. ~ $10^8 - 10^{11}\,L_\odot$

Mass to light ratio is 1 - 10 $\frac{M_\odot}{L_\odot}$

density of stars in gal is < 1% closure density.

Galaxies cluster ~ 50% are in groups of 1 - 10 large neighbours.

Correlation length for clusters is radius from one galaxy s.t. prob of finding another gal is 2 × prob if smoothly distributed. turns out to be ~ 5 Mpc.

~ 1% of bright gals occur in rich clusters > 30 members.

Clusters have ~ 25% of their mass as hot (10^{7-8} K) diffuse gas.

Bremsstrahlung radiates x-rays so can map it.

The speed of sound in the gas is similar to the speeds of the galaxies within the cluster ie ~ 500 - 1200 km s^{-1}

the temp agrees with the virial theorem.

Q.F.T. ①

$\hbar = c = 1$

Units + Dimensions

$[\text{speed}] = LT^{-1}$ ⇒ identify length with inverse time.

$[\text{ang momentum}]$

$= ML^2T^{-1}$

⇒ Identify : length $\equiv (\text{time})^{-1} \equiv (\text{mass})^{-1}$

ie $\boxed{L \equiv \frac{1}{T} \equiv \boxed{\frac{1}{M}}}$

So units of energy + momentum are mass

" of action are 1 (dimensionless)

energy × time

Simple Harmonic Oscillator.

Lagrangian : $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$ for m=

Canonical mom = $\dot{x}$ Hamiltonian $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2$

Quantise by postulating : $[x, p] = i$

Introduce ladder operators. $a = \frac{1}{\sqrt{2\omega}}(\omega x + ip)$ $a^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x - i$

Then $[a, a^\dagger] = 1$

also then $H = (a^\dagger a + \frac{1}{2})\omega$

→ a kills G.S. ⇒ GS energy $= \frac{1}{2}\omega$

if have ∞ osc then GS energy = ∞

If only interested in energy differences can use normal ordered form for H : $H = a^\dagger a\omega$. Now, G.S. has zero energy

Lorentz Invariance

A Lor. Inv f'n as it is $f(p^2)$

The "mass shell" δ-f'n is $\delta(p^2 - m^2)$

$f(p^2) = 0$ for 2 values of p....

Define $\delta_+(p^2 - m^2)$ which only contributes from $p = m$ not $p = -m$

now $p^2 - m^2 = (E^2 - |\underline{p}|^2) - m^2$
$= E^2 - E_p^2$

where $E_p = |\underline{p}|^2 + m^2$ (+ve root only)

$J = \det(L) = 1$

Now $\delta_+(p^2 - m^2)\,d^4p$ is a Lor Inv measure

Consider

$$I = \int g(p^0, \underline{p})\,\delta_+(E^2 - |\underline{p}|^2 - m^2)\,dE\,d^3\underline{p}$$

$$\Rightarrow I = \int \frac{g(E_p, \underline{p})}{2E_p}\,d^3\underline{p}$$ ← do the E integral

ie $\frac{d^3\underline{p}}{2E_p}$ is a Lorentz Invariant vol. meas.

Lorentz Transformations

$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$

now line element is preserved

$\Rightarrow \underline{\omega^{\mu\nu} = -\omega^{\nu\mu}}$

Consider infinitesimal L.T. :

$L^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

$\Rightarrow \omega^\mu{}_\nu =$ (time-space part is sym (space-space part is anti-sym))

time-space : Lorentz boosts

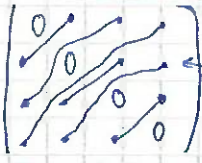
space-space : Spatial Rotations.

∃ 6 indep components of $\omega^\mu{}_\nu$ ∴ write as linear comb of 6 generators

$$\boxed{(M^{\rho\sigma})^\mu{}_\nu = g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu}$$

which matrix

which entry



6 matrices

Then, $\omega^\mu{}_\nu = \frac{1}{2}(\omega_{\rho\sigma}M^{\rho\sigma})^\mu{}_\nu$

number

ie coeff of matrix

matrix ie. $\cdot(\;) + \cdot(\;) + \cdot(\;)$ ... 6 time

The generators span the group's Lie Algebra space :

$$\boxed{\underbrace{[M^{\rho\sigma}, M^{\tau\upsilon}]}_{\text{matrix mult. on } \mu\nu \text{ indices.}} = g^{\sigma\tau}M^{\rho} - g^{\rho\tau}M^{\sigma\upsilon} + g^{\rho\upsilon}M^{\sigma\tau} - g^{\sigma\upsilon}M^{\rho\tau}}$$

in out   firsts lasts   out in   lasts firsts.

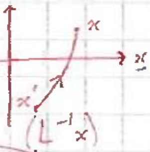
Finite Lor. Trans : $L = \exp(\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma})$

Lorentz Invariance of K-G Action

Want Lor. Inv. Action $S = \int d^4x\,\mathcal{L}(x)$

need $\int d^4x'\,\mathcal{L}(x')$
$= \int d^4x\,\mathcal{L}(x') =$

Lor. Trans :



o $\underline{\phi(x') = \phi'(x)}$

Scalar terms obviously transform like a scalar.

Derivative terms :

$\mathcal{L}'(x) = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi'$

but $\partial_\mu\phi'(x) = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma}\phi(x')$
$= (L^{-1})^\sigma{}_\mu\,\partial'_\sigma\phi(x')$

$$= \frac{1}{2}\underbrace{g^{\mu\nu}(L^{-1})^\sigma{}_\mu(L^{-1})^\tau{}_\nu}_{g^{\sigma\tau}}\partial'_\sigma\phi(x')\partial'_\tau\phi(x')$$

$= \mathcal{L}(x')$ → So $\mathcal{L}$ transforms like scalar.

For vector, trans. rule is $\boxed{A'^\mu(x) = L^\mu{}_\nu A^\nu(x')}$

→ Symmetry means that $\boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - X^\mu}$

∃ Conserved charge,

$Q = \int_{\text{space}} j^0\,d^3\underline{x}$, which satisfies : $\boxed{\frac{dQ}{dt} = 0}$ through $\partial_\mu j^\mu = 0$

Noether's Theorem

In a Lagrangian Field Theory, to every infinitesimal symmetry, there corresponds a conserved current j^μ s.t. $\partial_\mu j^\mu = 0$

Proof : let $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$

then $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha\Delta\phi\frac{\partial\mathcal{L}}{\partial\phi} + \underbrace{\alpha\Delta(\partial_\mu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}$ (Taylor Series)

$= \alpha\partial_\mu\left(\Delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \alpha\Delta\phi\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$

using product rule

$$\Rightarrow \mathcal{L}' = \mathcal{L} + \alpha\partial_\mu\left(\Delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$

$$\cancel{+ \alpha\Delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right)} \to 0$$ if E-L satisfied.

$$\text{So } \mathcal{L}' = \mathcal{L} + \alpha\partial_\mu\left(\Delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \underbrace{\partial_\mu X^\mu}_{\text{any other div term....}}$$

Q.F.T. ②

Complex K-G field

Applications of Noether's Theorem

① $\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{1}{2}m^2\phi^*\phi$

Sym is global phase rotation, $\phi \to \phi' = e^{i\alpha}\phi$

So $\alpha\Delta\phi = i\alpha\phi$ and $\alpha\Delta\phi^* = -i\alpha\phi^*$

$\Rightarrow j^\mu = i[(\partial_\mu\phi^*)\phi - \phi^*(\partial_\mu\phi)]$

can interpret as "electric" current, Q electric charge.

② Infinitesimal Translations

$x^\mu \to x'^\mu = x^\mu - a^\mu$

ie $(x-a)$

So $\phi(x) \to \underbrace{\phi'(x)}_{?} = \phi(x+a)$

$= \phi(x) + \overset{\Delta\phi_\nu}{\overline{a^\nu\partial_\nu\phi}}$

So $\partial_\mu\phi(x+a) = \partial_\mu\phi(x) + a^\nu\partial_\mu\underbrace{\partial_\nu\phi}_{\Delta\phi_\nu}$

$\mathcal{L}' = \mathcal{L} + \phi \text{ term} + \partial_\mu\phi \text{ term} + \underbrace{a^\nu\partial_\nu\mathcal{L}(x)}$

$= a^\nu\partial_\mu\underbrace{(\delta^\mu_\nu\mathcal{L})}_{X^\mu \text{ from before}}$

So current for each a^ν is

$$\boxed{T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}}$$

$\Rightarrow \partial_\mu T^\mu{}_\nu = 0$ from field eq'ns

(true for any $\mathcal{L}$ not exp. dep on x)

The Energy - Momentum Tensor

If $\nu = 0$, have time translation - get conserved energy:

$$E = \int T^{00}d^3\underline{x} = \int \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi - \mathcal{L}\,d^3\underline{x} = \int \underbrace{\pi\dot\phi - \mathcal{L}}_{H}\,d^3\underline{x}$$

If $\nu = 1 2 3$, have space translation - get conserved momentum:

$$p^i = \int T^{0i}d^3\underline{x} = \int \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^i\phi\,d^3\underline{x}$$

Classical Klein - Gordon Theory.

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \Rightarrow \underline{\partial^2\phi + m^2\phi = 0}$ K-G equation.

A solution is $\phi = e^{-ip\cdot x}$ ← 4 - vectors

$\Rightarrow E^2 = |\underline{p}|^2 + m^2$ - the dispersion relation for the waves

Another approach is to say that:

$\phi = a(t)e^{i\underline{p}\cdot\underline{x}}$. KG eq'n $\Rightarrow \underbrace{\ddot a + (|\underline{p}|^2 + m^2)a = 0}_{SHM}$

$\therefore \underline{a(t) = e^{\pm iE_p t}}$ - same result.

So we have an ∞ set of uncoupled oscillators.

General solution:

$$\phi(x) = \int d^3\underline{p}\left[f(p)e^{-ip\cdot x} + g(p)e^{ip\cdot x}\right] \quad \underline{\underline{\text{with } p^0 = E_p}}$$

only 3-mom integral because ie mass shell condition is implicit.

if ϕ real then $f^* = g$.

$H = \int d^3\underline{x}\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\underline{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right)$

$\underline{P} = -\int \pi\,\underline{\nabla}\phi\,d^3\underline{x}$

So $\underbrace{H = \int\frac{d^3\underline{p}}{(2\pi)^3}E_p a^\dagger_{\underline{p}} a_{\underline{p}}}_{\text{have normal ordered}}$ $\quad \underline{P} = \int\frac{d^3\underline{p}}{(2\pi)^3}\underline{p}\,a^\dagger_{\underline{p}}a_{\underline{p}}$

coz only energy diff. important.

Now $[H, a_p^\dagger] = E_p a_p^\dagger$

$[H, a_p] = -E_p a_p$

→ raises/lowers energy of any state by E_p.

Start from vacuum state: $\underline{a_p|0\rangle = 0}$

Similarly, $\underline{P}$ raises/lowers momentum by $\underline{p}$.

Also $[H, \underline{P}] = \underline{0}$ So can have sim. e-states

Normalisation: $\langle 0|0\rangle = 1$

define $|p\rangle$ s.t. $\langle\underline{p}|\underline{q}\rangle = \underbrace{2E_{\underline{p}}}_{\text{for Lor. Inv}}(2\pi)^3\delta^{(3)}(\underline{p}-\underline{q})$

ie $p \to p'$ $q \to q'$ still have same normalisation.

$|p\rangle = \sqrt{2E_p}\,a^\dagger|0\rangle$ [So $H|p\rangle = E_p|p\rangle$, $\underline{P}|p\rangle = \underline{p}|p\rangle$]

Quantised Klein-Gordon Theory.

Postulate $[\phi(\underline{x}), \pi(\underline{x}')] = i\delta^{(3)}(\underline{x}-\underline{x}')$ $\quad [\phi(\underline{x}), \phi(\underline{x}')] = 0$

$[\pi(\underline{x}), \pi(\underline{x}')] = 0$

EQUAL-TIME COMMUTATION RELATIONS

Express in terms of creation and anihilation operators:

$$\phi(\underline{x}) = \int\frac{d^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\left(a_{\underline{p}}e^{i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^\dagger e^{-i\underline{p}\cdot\underline{x}}\right)$$

$$\Rightarrow \pi(\underline{x}) = \int\frac{d^3\underline{p}}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left(a_p e^{i\underline{p}\cdot\underline{x}} - a_p^\dagger e^{-i\underline{p}\cdot\underline{x}}\right)$$

time dep. is implicit $e^{\pm iE_p t}$.

So:

$[a_{\underline{p}}, a_{\underline{p}'}^\dagger] = (2\pi)^3\delta^{(3)}(\underline{p}-\underline{p}')$ $\quad [a_{\underline{p}}, a_{\underline{p}'}] = 0$

$[a_{\underline{p}}^\dagger, a_{\underline{p}'}^\dagger] = 0$

these are equivalent to these

Now:

$$\phi|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-i\underline{p}\cdot\underline{x}}|\underline{p}\rangle$$

ie almost a "particle at x" state (if no $\frac{1}{2E_p}$) ie a wave-packet with forier coeffs to give f-f'n

So could interpret ϕ "creating particle at x"

But:

ϕ also contains anihilation operators: for eg. consider ϕ acting on a one-particle state $|p\rangle$:

$\phi|p\rangle = $ two particle piece $|p';p\rangle$ + coeff. $|0\rangle$ — zero particle piece.

coeff = $\boxed{\langle 0|\phi|p\rangle = e^{i\underline{p}\cdot\underline{x}}}$ Projection of $\phi|p\rangle$ onto the vacuum.

ie $\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}e^{i\underline{q}\cdot\underline{x}}\langle 0|a_q|p\rangle$ → $\sqrt{2E_p}\,a^\dagger|0\rangle$

$\langle 0|a_q a_p^\dagger|0\rangle = \cancel{\langle 0|a_p^\dagger a_q|0\rangle} + \langle 0|[a_q, a_p^\dagger]|0\rangle$

annihilation

$(2\pi)^3\delta(\underline{p}-\underline{q})$

Q.F.T. ③

Heisenberg Picture

- Have used S.P. until now where the states themselves have a time dependence: e^{-iEt} ← total energy

In H.P. operators obey:

$$i\frac{d}{dt}\hat{O}(t) = [\hat{O},\hat{H}]$$

Equivalently, $\hat{O}(S.P.)$ is conjugated by $e^{\pm i\hat{H}t}$:

$$\hat{O}(t) = e^{i\hat{H}t}\,\hat{O}(t=0)\,e^{-i\hat{H}t}$$

NB - need to know $\hat{H}$ in order to get H.P. fields - tricky eg I.P.!

→ Consider K-G field: we have

$$Ha_p = a_p(H-E_p)$$

$$\Rightarrow H^n a_p = a_p(H-E_p)^n$$

$$\Rightarrow \boxed{e^{iHt}a_p e^{-iHt} = a_p e^{-iE_pt}}$$

Similarly for $a_p^\dagger$....

So $\phi(\underline{x})$ becomes

$$\phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

4 - vectors

Time translations ↙

Space translations:

$$\underline{P}a_p = a_p(\underline{P}-\underline{p})$$

$$\text{So } \phi(\underline{x}) = e^{i\underline{P}\cdot\underline{x}}\,\phi(\underline{0})\,e^{-i\underline{P}\cdot\underline{x}}$$

→ define $e^{iP\cdot x} = e^{i(Ht - \underline{P}\cdot\underline{x})}$ H and $\underline{P}$ commute so OK

So then can write both H and $\underline{P}$ conj's as

$$\underline{\phi(x) = e^{iP\cdot x}\,\phi(0)\,e^{-iP\cdot x}}$$

Again, need to know all e-states of $\underline{P}$.

Propagator for K-G field.

Split up ϕ as $\phi^+(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p e^{-ip\cdot x}$

+ve frequency, destruction

−ve freq, creation

and $\phi^-(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p^\dagger e^{ip\cdot x}$

$[\phi^+,\phi^+]=0$ and $[\phi^-,\phi^-]=0$

Now:

$$[\phi(x),\phi(y)] = [\phi^+(x),\phi^-(y)] + [\phi^-(x),\phi^+(y)]$$

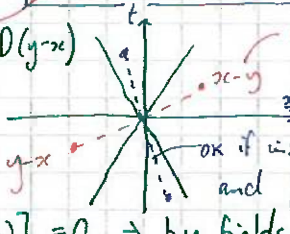
$$= \boxed{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}} - \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(y-x)}$$

Lorentz Invariant f'n of $(x-y)$

$$= \boxed{D(x-y) - D(y-x)}$$

$D(x-y) - D(y-x) = 0$ if $x-y$ spacelike

ie. ↓



can transform $x-y$ into $y-x$ if outside light cone

ok if inside - ie. future and past not "connected"....

$[\phi(x),\phi(y)] = 0 \Rightarrow$ two fields are simultaneously knowable if cannot talk to each other....

Now take v.e.v. of $\phi(x)\phi(y)$:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|[\phi^+(x),\phi^-(y)]|0\rangle$$

using $\phi^+|0\rangle = 0$ and $\langle 0|\phi^- = 0$

$$= D(x-y)\langle 0|0\rangle$$

$$= \underline{D(x-y)}$$

So this is the amplitude for a particle to be created at y then to disappear at x ie to propagate from $y \to x$.

But $y^0 > x^0$ or $x^0 > y^0$ ..... let's use:

Feynman prop: $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$

$$\text{So } \boxed{D_F(x-y) = \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)}$$

Now also,

$$\boxed{D_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}}$$

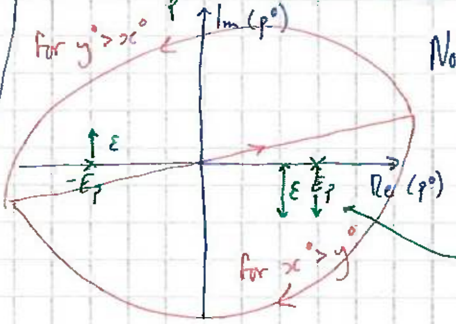
Because

$$\frac{1}{p^2-m^2} = \frac{1}{(E^2-E_p^2)}$$

E ← p^0

So do contour integration with contours

(∃ others... ret/adv)



Note - D_F is G.F. for K-G operator and shorthand notation for contour shift is:

$$D_F = \int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon}$$

Classical Dirac Theory

Dirac Equation: $\boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0}$

Seems Lor inv coz space + time equiv. in it ie both First order. But Lor Inv $\Rightarrow \gamma^\mu$ but is a matrix.... (D) should ⇒ K-G:

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0$$

$$\Rightarrow (-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2)\psi(x) = 0$$

$$\text{So get K-G if } \boxed{\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}I}$$

(Only sym. part)   ie $\underline{\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}}$

Each of the 4 components of ψ satisfies K-G but (D) is stronger - mixes the components up.

This is called the Dirac Algebra

→ Simplest rep of Dirac Algebra is with 4×4 matrices (n×m wi.do

Choose them as: $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}$

$\sigma^1 = \begin{matrix} 0 & 1 \\ 1 & 0\end{matrix}$

$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

Can also take any equivalent set by rotating basis spinors: $\psi \to U\psi$ and $\gamma^\mu \to U\gamma^\mu U^{-1}$ for any unit. invertable U

then this is still satisfied.

Let $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho,\gamma^\sigma]$ then $S^{\rho\sigma}$ gives a representation of the Lorentz algebra:

$$\boxed{[S^{\rho\sigma},S^{\tau\nu}] = g^{\sigma\tau}S^{\rho\nu} - g^{\rho\tau}S^{\sigma\nu} + g^{\rho\nu}S^{\sigma\tau} - g^{\sigma\nu}S^{\rho\tau}}$$

$M^{\rho\sigma}$ and $S^{\rho\sigma}$ are both 4×4 matrices but they are different, inequivalent representations of the Lorentz generators

This would be more clear if space time had a different dimension say 6 - then $M^{\rho\sigma}$ is 6×6 but $S^{\rho\sigma}$ is 8×8

Q.F.T. ④

Lorentz Invariance of Dirac Equation

Under commutation with $S^{\rho\sigma}$, γ^μ transforms like a Lorentz 4-vector:

$$[\gamma^\mu, S^{\rho\sigma}] = \underbrace{(M^{\rho\sigma})^\mu{}_\nu}_{\text{this is a Lor. trans.}} \gamma^\nu$$

$\Rightarrow \ldots$

$$\left(1-\tfrac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1+\tfrac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)$$

$$= \left(1+\tfrac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu \gamma^\nu$$

this is the inf'l form of conj: $S(L)^{-1}\gamma^\mu S(L)$

Now - have Lorentz Trans.

$$L = \exp\left(\tfrac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)$$

and have spinor rep:

$$S(L) = \exp\left(\tfrac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)$$

Postulate spinor trans. law:

$$\psi'_\alpha(x) = S(L)_{\alpha\beta}\psi_\beta(x')$$

spinor indices

where $x' = L^{-1}x$

Now, does ψ' still obey the Dirac eq'n?

$$(i\gamma^\mu\partial_\mu - m)\psi'$$

$$= (i\gamma^\mu\partial_\mu - m)S(L)\psi(x')$$

$$= S(L)\left[iS(L)^{-1}\gamma^\mu S(L)\partial_\mu - m)\right]\psi(x')$$

$$= S(L)\left[iL^\mu{}_\nu\gamma^\nu\partial_\mu - m\right]\psi(x')$$

$$= S(L)\left[iL^\mu{}_\nu\gamma^\nu(L^{-1})^\lambda{}_\mu\partial'_\lambda - m\right]\psi(x')$$

$$= S(L)(i\gamma^\lambda\partial'_\lambda - m)\psi(x')$$

as ordering of L, γ no matter - just numbers.

$= 0$ So transformed field also is a solution of (D)

Hermiticity Properties (Conjugation)

Dirac algebra $\Rightarrow (\gamma^0)^2 = I$, $(\gamma^i)^2 = -I$

let γ^0 be Herm. γ^i can be anti herm.

$$\underline{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu(\gamma^0)^{-1}}$$ expresses the herm. properties

So $(S^{\rho\sigma})^\dagger = -\gamma^0 S^{\rho\sigma}(\gamma^0)^{-1}$

So $S(L)^\dagger = \gamma^0 S(L)^{-1}(\gamma^0)^{-1}$

ie bring conj. outside the exp.

$\to S(L)$ not a unitary rep as Lor. group not compact... but very close.

DIRAC CONJUGATE : $\bar\psi(x) = \psi^\dagger(x)\gamma$

$\to$ allows us to construct tensor densities:

scalar $s(x) = \bar\psi(x)\psi(x)$

vector current $j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)$

tensor density $t^{\mu\nu}(x) = \bar\psi S^{\mu\nu}\psi$.

Lor Trans: $\psi \to S(L)\psi(x')$

$$\psi^\dagger \to \psi^\dagger(x')S(L)^\dagger$$

$$\text{So } \bar\psi \to \psi^\dagger(x')\gamma^0 S(L)^{-1}$$

$\Rightarrow$ $\underline{s(x) \to s(x')}$ scal. density - transforms as a scalar field

Dirac Lagrangian, Currents, Norms

$$S = \int d^4x \left(\bar\psi(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\right)$$

then (D) is $\boxed{\dfrac{\partial\mathcal{L}}{\partial\bar\psi} = 0}$

Conjugate eq'n $\dfrac{\partial\mathcal{L}}{\partial\psi} = 0$ gives:

$$i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi = 0$$

Now, the current $\underline{j^\mu = \bar\psi\gamma^\mu\psi}$ is conserved (use ↑)

Can derive from Noether's theorem: sym is $\psi \to e^{i\alpha}\psi$.

$\exists$ a conserved charge, the Norm:

$$N = \int \bar\psi\gamma^0\psi\, d^3\underline{x} = \int \psi^\dagger\psi\, d^3\underline{x} \geq 0$$

Now, $\underline{\gamma^5} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$

Basic properties: ① $\{\gamma^5, \gamma^\mu\} = 0$

and ② $(\gamma^5)^2 = I$

(γ^5 is parity odd ....)

now $[\gamma^5, S^{\mu\nu}] = 0$ So γ^5 is Lorentz Invariant.

Can construct more tensor densities: $p(x) = \bar\psi\gamma^5\psi$ pseudoscalar density.

and $a^\mu(x) = \bar\psi\gamma^\mu\gamma^5\psi$ - axial vector current.

Use as before to show that

$$\partial_\mu a^\mu = 2im\bar\psi\gamma^5\psi$$

So a conserved in the massless case eg neutrinos - $\underline{\underline{2}}$ currents.

Chiral Spinors

- e-vecs of γ^5:

$$\gamma^5\begin{pmatrix}\chi\\ \chi\end{pmatrix} = +1\begin{pmatrix}\chi\\ \chi\end{pmatrix}$$ Right handed

$$\gamma^5\begin{pmatrix}\chi\\ -\chi\end{pmatrix} = -1\begin{pmatrix}\chi\\ -\chi\end{pmatrix}$$ Left handed

When $m=0$, can get pure left or right handed sol'ns to (D). But when $m \neq 0$, they are mixed.

- have reduced spinor rep.

Solutions of the Dirac Equation

Plane waves: try $\psi(x) = u(p)e^{-ip\cdot x}$

$+^{ve}$ energy $E = E_p$

$-^{ve}$ energy sol'ns, $E = -E_p$ $\psi(x) = v(p)e^{ip\cdot x}$

$$\Rightarrow \boxed{\begin{array}{c}(\gamma^\mu p_\mu - m)u(p) = 0\\ (\gamma^\mu p_\mu + m)v(p) = 0\end{array}}$$

For $E = E_p$, let $u(p) = \begin{pmatrix}\chi\\ \emptyset\end{pmatrix}$ then get $u(p) = \sqrt{E+m}\begin{pmatrix}\chi\\ \dfrac{\underline{\sigma}\cdot\underline{p}\,\chi}{E+m}\end{pmatrix}$ from

const 2-spinor

A basis for is χ_s for $s = -\tfrac{1}{2}, \tfrac{1}{2}$

where $\chi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$. A complete set of $+^{ve}$ energy solutions is:

$$\left\{u_s(p)e^{-ip\cdot x} = \sqrt{E_p+m}\begin{pmatrix}\chi_s\\ \dfrac{\underline{\sigma}\cdot\underline{p}\,\chi_s}{E+m}\end{pmatrix}e^{-ip\cdot x}\right\}_{s=-\frac{1}{2},\frac{1}{2}}$$

If zero mom get $u_s = \sqrt{2m}\begin{pmatrix}\chi_s\\ 0\end{pmatrix}e^{-imt}$ then Lor. boost gives back this eq'n

Sim'ly, $-^{ve}$ freq. solutions are:

$$\psi_-(x) = \sum_s v_s(p)e^{ip\cdot x} = \sum_s \sqrt{E+m}\begin{pmatrix}\dfrac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi_s\\ \chi_s\end{pmatrix}e^{ip\cdot x}$$

Q.F.T. ⑤

Spin sums and Projection Operators

$\bar{u}u$ is scalar, $u\bar{u}$ is exterior product - 4×4 matrix.

From the explicit forms of $u_s(p)$, $v_s(p)$ we have:

$$\boxed{\bar{u}_s(p)\, u_{s'}(p) = 2m\,\delta_{ss'}} \quad \boxed{\bar{v}_s(p)\, v_{s'}(p) = -2m\,\delta_{ss'}}$$

$$\text{and} \quad \boxed{\bar{u}_s(p)\, v_{s'}(p) = 0}$$

From the momentum - Dirac eq'ns we have:

$$\begin{cases}(\gamma\cdot p + m)\, u_s(p) = 2m\, u_s(p) \\ (\gamma\cdot p - m)\, v_s(p) = -2m\, v_s(p)\end{cases}$$

$$\Rightarrow \sum_s u_s(p)\,\bar{u}_s(p) = \gamma\cdot p + m$$

$$\sum_s v_s(p)\,\bar{v}_s(p) = \gamma\cdot p - m$$

to verify act on $u_{s'}(p)$ or $v_{s'}(p)$ and use

$\frac{1}{2m}(\gamma\cdot p + m)$ is a projection op. onto $+^{ve}$ en. sol'ns

$-\frac{1}{2m}(\gamma\cdot p - m)$ is a projection op. onto $-^{ve}$ energy sol'ns.

Quantisation of the Dirac Field.

$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$. Field conjugate to Ψ,

$$\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\Psi)}\right)_\alpha = i(\bar{\Psi}\gamma^0)_\alpha = i\Psi^\dagger_\alpha = \xi_\alpha$$

Postulate equal time canonical anticommutation relations:

$$\{\Psi_\alpha(\underline{x}), i\Psi^\dagger_\beta(\underline{x}')\} = i\,\delta_{\alpha\beta}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

cancel.

$\times\gamma^0$ if want to write using $\bar{\Psi}$.

$$\text{and } \{\Psi_\alpha(\underline{x}), \Psi_\beta(\underline{x}')\} = \{\bar{\Psi}_\alpha(\underline{x}), \bar{\Psi}_\beta(\underline{x}')\} = 0.$$

Hamiltonian $H = \int d^3\underline{x}\,(\xi_\alpha(\underline{x})\,\dot{\Psi}_\alpha(x) - \mathcal{L})$

time dependence cancels out

$$\Rightarrow H = \int d^3\underline{x}\,(i\bar{\Psi}(\underline{x})\,\underline{\gamma}\cdot\underline{\nabla}\,\Psi(\underline{x}) + m\bar{\Psi}(\underline{x})\Psi(\underline{x}))$$

In a plane wave basis, the Dirac field is: (SP)

$$\Psi(\underline{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a^s_{\underline{p}}\, u_s(p)\, e^{i\underline{p}\cdot\underline{x}} + b^{s\dagger}_{\underline{p}}\, v_s(p)\, e^{-i\underline{p}\cdot\underline{x}}\right)$$

The Dirac Conjugate field is:

$$\Psi(\underline{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a^{s\dagger}_{\underline{p}}\,\bar{u}_s(p)\, e^{-i\underline{p}\cdot\underline{x}} + b^s_{\underline{p}}\,\bar{v}_s(p)\, e^{i\underline{p}\cdot\underline{x}}\right)$$

Anticom. rels become:

$$\{a^s_{\underline{p}}, a^{s'\dagger}_{\underline{p}'}\} = \{b^s_p, b^{s'\dagger}_{\underline{p}'}\} = (2\pi)^3\,\delta^{(3)}(\underline{p}-\underline{p}')\,\delta^{ss'}$$

$$\text{and } \{a,a\} = \{b,b\} = \{a^\dagger,a^\dagger\} = \{b^\dagger,b^\dagger\} = 0.$$

Hamiltonian becomes: $\boxed{H = \int\frac{d^3\underline{p}}{(2\pi)^3}\sum_s E_p\,(a^{s\dagger}_p a^s_p - b^s_p b^{s\dagger}_p)}$

Write in normal ordered form ⇒ must discard an ∞ constant - no prob. Const is $\underline{\underline{-^{ve}}}$ coz anti com rels!

now $H a^{s\dagger}_p = a^{s\dagger}_p(H + E_p) \qquad H a^s_p = a^s_p(H - E_p)$

and $H b^{s\dagger}_p = b^{s\dagger}_p(H + E_p) \qquad H b^s_p = b^s_p(H - E_p)$

So $a^\dagger$, $b^\dagger$ are creation operators for two different types of particle, but both with $+^{ve}$ energy E_p.

a one particle state is $|p,s\rangle = \sqrt{2E_p}\, a^{s\dagger}_p|0\rangle$

a diff. one is $|p,s\rangle' = \sqrt{2E_p}\, b^{s\dagger}_p|0\rangle$

a two particle state is: $|p_1 s_1, p_2 s_2\rangle$

$$= \sqrt{4E_{p_1}E_{p_2}}\; a^{s_1\dagger}_{p_1}\, a^{s_2\dagger}_{p_2} = -|p_2 s_2, p_1 s_1\rangle$$

hence Fermi - Dirac Statistics.

Heisenberg Picture Dirac field:

$$\Psi(x) = \int\frac{d^3\underline{p}}{(2\pi)^3\sqrt{2E_p}}\sum_s\left(a^s_{\underline{p}}\, u_s(p)\, e^{-ip\cdot x} + b^{s\dagger}_p\, v_s(p)\, e^{ip\cdot x}\right)$$

and it solves $i\frac{\partial\Psi}{\partial t} = [\Psi, H]$ (ie. (D) !)

Propagator for Dirac Field.

$iS(x-y) = \{\Psi(x), \bar{\Psi}(y)\}$ no sum on spinor indices ie $\{\Psi_\alpha, \Psi_\beta\}$

$$\Rightarrow iS(x-y) = (i\gamma\cdot\partial_x + m)\,(D(x-y) - D(y-x))$$

Now $\langle 0|\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle$ gives ie only a and $a^\dagger$ contribute

and $\langle 0|\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle$ gives ie. only b and $b^\dagger$ contribute.

Feynman Prop: $iS_F(x-y) = \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle$ where:

$$T\Psi_\alpha(x)\bar{\Psi}_\beta(y) = \Psi_\alpha(x)\bar{\Psi}_\beta(y) \quad \text{if } x^0 > y^0$$

$$= -\bar{\Psi}_\beta(y)\Psi_\alpha(x) \quad \text{if } x^0 < y^0.$$

$-^{ve}$ sign so continuous as y^0 increases through x^0

Also, $(i\gamma\cdot\partial_x - m)\, S_F(x-y) = i\,\delta^4(x-y)$

ie $S_F(x-y)$ is ~~the~~ a G.F. for Dirac operator.

iS_F has the integral representation:

$$\boxed{iS_F(x-y) = -\int\frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\,\frac{\gamma\cdot p + m}{p^2 - m^2 + i\varepsilon}}$$

Formally we can write:

$$iS_F(x-y) = -\int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{(\gamma\cdot p - m + i\varepsilon)}$$

the "momentum form" of the Dirac operator.

Particles and Antiparticles

Remember in classical Dirac theory $N = \int \psi^\dagger \psi \, d^3x$ and $\dot{N} = 0$.

When couple Dirac field ~~is~~ to e/m gauge potential, a_μ, get term in Lag. : $e\, a_\mu(x)\, j^\mu(x)$

where $j^\mu = \bar{\psi} \gamma^\mu \psi$. The term $e a_0 j^0 = e a_0 \psi^\dagger \psi \Rightarrow e \psi^\dagger \psi$ is the <u>electric</u> charge density of the Dirac field.

→ eN is then the charge operator (after throw away $-\infty$)

Now $eN = e \int \frac{d^3p}{(2\pi)^3} \sum_s (\underbrace{a_p^{s\dagger} a_p^s}_{\text{number op for a type}} - \underbrace{b_p^{s\dagger} b_p^s}_{\text{for b type}})$

Now, $eN|0\rangle = 0$ and :

$$\boxed{\begin{array}{l} a_p^{s\dagger} \text{ creates particles of charge } e \\ b_p^{s\dagger} \text{ creates particles of charge } -e \end{array}}$$

(from a/com rels)

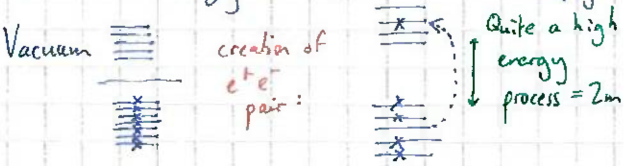
We have a theory of free electrons and positrons.

N counts (number of e^- — number of e^+)

e^+, e^- <u>same</u> except charge + magnetic mom

Dirac's Hole Interpretation

Orig. idea was theory of one particle type with +ve + −ve energy states, either occ. or empty.



Interacting Q.F.T. - Overview

Quadratic Lag. ⇒ linear field eq'ns

⇒ superposition ⇔ non-interaction.

Interactions : higher order terms → non linear equations.

Eg: (1) <u>Scalar ϕ^4 theory</u>:

$$\mathcal{L} = \mathcal{L}_{K.G.} + \frac{\mu}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

(2) <u>Yukawa Theory</u>

$$\mathcal{L} = \mathcal{L}_{K.G.} + \mathcal{L}_{Dirac}$$

(3) QED. $\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{Dirac} + \mathcal{L}_{int}$ where $\mathcal{L}_{int} = -e \bar{\psi} \gamma^\mu \psi a_\mu$.

Int. terms always <u>at a point</u> so causility OK.

<u>Renormalisation</u>: Dimensions of $\mathcal{L}$ is $[M]^4$. So can get dims of <u>coupling</u> consts eg λ is dim'less. If −ve power then coupling is <u>non</u> renormalisable... because : loop terms create/dest. particles with arb. large mom. When integrate over mom → diverge. So introduce momentum cut off or lattice spacing $\frac{1}{\Lambda}$. Integrals depend on Λ via $\frac{m}{\Lambda}$ or $\frac{\mu}{\Lambda}$ etc...

<u>Then</u> can sometimes carefully take $\lim \Lambda \to \infty$.

With non-renorm. couplings, results go as <u>+ve</u> <u>powers</u> of Λ → probs.

The Interaction Picture

I.P. is H.P. but using H_0....

ie let $H = H_0 + H_{int}$

then $\phi_{I.P.}(x) = e^{iH_0 t} \phi_{S.P.}(\underline{x}) e^{-iH_0 t}$

Now S.P. : S.E. is $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$

time dep. Schr. states

diff. time dep.

let I.P. <u>states</u> be $|\varphi\rangle$ where $|\psi\rangle = e^{-iH_0 t}|\varphi\rangle$

f'n of t

diff dep. on t.

now, subst into S.E. to get :

$$i\frac{d}{dt}|\varphi\rangle = \underbrace{e^{iH_0 t} H_{int} e^{-iH_0 t}}_{H_I} |\varphi\rangle$$

The interaction part of the Ham. in the interaction picture

Now in scalar ϕ^4 theory with $\mu = 0$, $H_{int} = \int d^3\underline{x} \frac{\lambda^4}{4!} \phi^4(x)$

$$\Rightarrow \boxed{H_I = \int d^3\underline{x} \frac{\lambda^4}{4!} \phi^4_{I.P.}(x)}$$

using $e^{iH_0 t}\phi e^{-iH_0 t} e^{iH_0 t} \phi e^{-iH_0 t}$ <u>etc</u>

S-Matrix - time evolution operator

Can iteratively solve this equation $i\frac{d}{dt}|\varphi\rangle = H_I|\varphi\rangle$

let $U(t)$ be the time evolution operator, s.t. $|\varphi\rangle = U(t)|i\rangle$ then, sol'n is :

$$U(t) = 1 + (-i)\int_{-\infty}^{t} dt' \, H_I(t') U(t')$$

Now subst to get:

$$U(t) = 1 + (-i)\int_{-\infty}^{t} dt' H_I(t') + (-i)^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' H_I(t') H_I($$

$+ \ldots$

here, $t'' < t'$

Now let $S(t)$ be :

$$\boxed{S(t) = \lim_{t\to\infty} U(t)}$$

here, either $2 \times \frac{1}{2}$

Using the T-product, can write as exp. series: ie. must divide by <u>n!</u> if integrate n times

So :

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \ldots dt_n \, T\left[H_I(t_1) \cdots H_I(t_n)\right]$$

or in short hand, $S = T e^{-i\int_{-\infty}^{\infty} H_I(t) dt}$

Wick's Theorem

In S-matrix elements ∃ T-products. To calc v.e.v.'s we need normal ordered products.

$$\boxed{T\phi(x_1)\phi(x_2)\cdots\phi(x_m) = :\phi_1\phi_2\cdots\phi_m: + \ :\text{all contractions}:}$$

To prove, split fields into $\phi^+ + \phi^-$, commute past each other for each time ordering case eg $x_1^0 > x_2^0$ or $x_1^0 < x_2^0$ etc...

For 2 fields : let $x^0 > y^0$ then $T\phi_x\phi_y = \phi_x\phi_y$

$$= (\phi_x^+ + \phi_x^-)(\phi_y^+ + \phi_y^-)$$

this term swapped using

$$= \underbrace{\phi_x^+\phi_y^+ + \phi_x^-\phi_y^+ + \phi_x^-\phi_y^-}_{\text{already normal ordered ie } \phi^+ \text{ to right.}} + \phi_y^-\phi_x^+ + [\phi_x^+, \phi_y^-]$$

Similarly if $y^0 > x^0$, get $[\phi_y^+, \phi_x^-]$

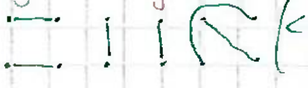
So $T\phi_x\phi_y = :\phi_x\phi_y: + D_F(x-y)$

2 important theorems :

① $\langle 0|T\phi_1\cdots\phi_m|0\rangle = 0$ if m is odd.

② $\langle 0|T\phi_1\cdots\phi_m|0\rangle = \sum_{\text{all poss pairs}} D_F(x_{i_1}-x_{i_2})(x_{i_3}-x_{i_4})\cdots\cdots D_F(x_{i_{m-1}}-x$

↳ Diagramatically for $\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle$ get:



($\langle 0|$ $|0\rangle$ kills all the rest...)

QF.T. ⑦

Wick's Theorem for terms like: $<0|T(\phi_1 \cdots \phi_m S)|0>$ in scalar ϕ^4 theory

ϕ^4 theory $\Rightarrow S = \sum_n \left(\frac{-i\lambda}{4!}\right)^n \frac{1}{n!}\int_{M^4} T\left[\phi^4(x_1)\cdots\phi^4(x_n)\right] d^4x_1 \cdots d^4x_n$

So the nth order term is

$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int_{M^4} <0|T(\phi_1 \cdots \phi_m \phi_1^4 \cdots \phi_n^4)|0> d^4x_1 \cdots d^4x_n$

Wick $\Rightarrow$ only terms where all fields are contracted survive after do: $<0|\quad|0>$.

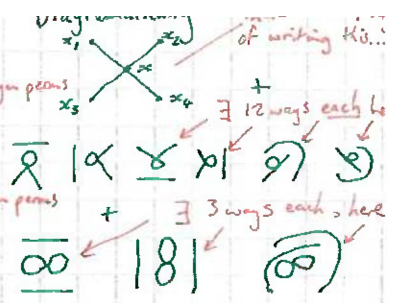
eg: 4 external fields and 1st order S-matrix:

$\frac{-i\lambda}{4!}\int d^4x <0|T(\phi_1\phi_2\phi_3\phi_4 \times \phi^4(x)|0>$

$= \phi_1\phi_2\phi_3\phi_4\phi\phi\phi\phi(x)$ + no perms + sym perms

$+ \phi_1\phi_2\phi_3\phi_4\phi\phi\phi\phi(x)$ + 5 perms + sym perms

$+ \phi_1\phi_2\phi_3\phi_4\phi\phi\phi\phi(x)$ + 2 perms + sym perms.



$= \frac{-i\lambda}{\cancel{4!}} \cancel{4!} \int D_F(x_1-x)D_F(x_2-x)D_F(x_3-x)D_F(x_4-x)\,d^4x$ + no perms

$+ \frac{-i\lambda}{\cancel{4!}\,2} \cancel{12} \int D_F(x_3-x)D_F(x_4-x)D_F(0)\,d^4x \,.\, D_F(x_1-x_2)$ + 5 perms

$+ \frac{-i\lambda}{\cancel{4!}\,8} \cancel{3} \int D_F(0)D_F(0)\,d^4x \; D_F(x_1-x_2)D_F(x_3-x_4)$ + 2 perms

The symmetry permutations (combinations) work out to be sym. factors which we divide by

eg 8 can exchange each loop end + loops themselves

another eg: 4 "external" fields and 2nd order S-matrix

take the completely connected terms (3 of them) like:

$\frac{(-i\lambda)^2}{2}\int D_F(x_1-x)D_F(x_2-x)D_F(x_3-y)D_F(x_4-y)D_F^2(x-y)\,d^4x\,d^4y$

from $\phi_1\phi_2\phi_3\phi_4\,\phi(x)\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_y$

12 ways for 1, 2 with x

12 ways for 3, 4 with y

4 ways x with y

$\frac{12 \times 12 \times 2 \times 2}{4! \times 4!} = \frac{1}{2}$

diagramatically:

ie for each line we have propagator

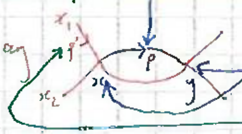
and for each vertex we have $-i\lambda\int d^4x$

then don't forget symmetry factor ()

Position space Feynman rules

But! $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-y)}$

ie say



Then at say x vertex, $\int$ over x – exponentials give δ-f'ns which are equivalent to imposing 4-momentum conservation.

So in momentum space, for each external point have $e^{-ip\cdot x}$, say

at each vertex, have ($-i\lambda$ + momentum conservation) and

for each internal line, have $\frac{i}{p^2-m^2+i\varepsilon}$ and $\int$ over internal momenta

Vacuum Bubbles

Remarkably, the symm. factors work out such that: $<0|S|0>$ is... $(= <0|\phi^4\text{ terms}|0>)$

$<0|S|0> = \exp\left(\text{all distinct vac. bubbles}\right)$

eg

etc ie $e^{8} = 1 + 8 + \frac{88}{2!} + \dots$

Now $<0|\phi_1 \cdots \phi_m S|0> = \sum$ diagrams with m external points

$= \left(\sum \text{connected diagrams}\right) \times \exp\left(\text{all distinct vac. bubbles}\right)$

others are killed by normal ordering

quite remarkable

eg

$<\Omega|T\phi_1\cdots\phi_m(S)|\Omega> \quad <0|S|0>$

ie not nec, tree

vacuum of interacting theory

S evolves non-int vacuum into all poss. <0| then

Can leave S in if want.... true vacuum doesn't evolve so $S|\Omega> = |\Omega>$ gives correction

If divide by correction, O.K. [NB $<\Omega|\Omega> = 1$]

ie only keep connected diagrams

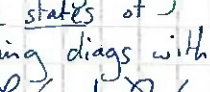
2 particle elastic scattering in scalar ϕ^4 theory

ie $<p_3, p_4|S|p_1, p_2>$

need to contract $\phi(x)|p_1> = e^{-ip_1\cdot x}$ (see QF.T ②)

now $|p_1, p_2>$ is a state of the non-interacting theory

Non-interacting vacuum was → correct one by only using connected diags. Now, can change states of non-int theory into correct ones by ignoring diags with loops attached to external lines eg



So for each external line – an incoming particle for eg, the loops change the mass to the observed value – the particle interacts with the vac taking it off mass shell – $p^2 \neq m^2$.

(LSZ perscription)

now take for eg, a second order term:

$<p_3, p_4|\frac{(-i\lambda)^2}{(4!)^2}\iint d^4x\,d^4y\,\phi_x\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_y|p_1, p_2>$

$= \frac{(-i\lambda)^2}{2}\iint d^4x\,d^4y \overbrace{\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-y)}}^{D_F(x-y)} \overbrace{\int \frac{d^4p'}{(2\pi)^4}\frac{i}{p'^2-m^2+i\varepsilon}e^{-ip'\cdot(x-y)}}^{D_F(x-y)} e^{i(p_3+p_4)\cdot x} \times e^{-i(p_1+p_2)\cdot y}$

sym → 2

x and y integrals give $\delta^{(4)}(p_3+p_4-p-p')\,\delta^{(4)}(p_1+p_2-p-p')$

then dp' integral sets $p' = p + p - p$ say ....

$= \frac{(-i\lambda)^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}\frac{i}{(p+p_1+p_2)^2-m^2+i\varepsilon} \times \delta^{(4)}(p_3+p_4-p_1\cancel{+}-p_2) \equiv$

So rules:

external line $e^{ip_x\cdot x}$

vertex – $(-i\lambda)$ then $\int d^4x$

line – propagator D_F

Position Space

external line – 1

vertex – $(-i\lambda)$ + mom cons.

line – $\frac{i}{p^2-m^2+i\varepsilon}$ + $\int d^4p$

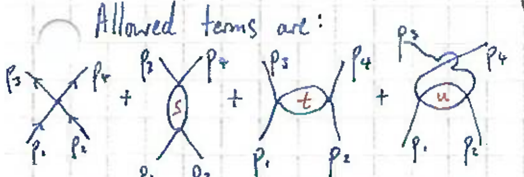
overall mom/energy cons.

momentum space

Q.F.T. ⑧

2 particle ϕ^4 continued

Allowed terms are:



1st order — 2nd order

s channel — t channel — u channel

$s=(p_1+p_2)^2$ $t=(p_1-p_3)^2$ $u=(p_1-p_4)^2$

$-i\lambda$

$$\hookrightarrow \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k+p_1+p_2)^2-m^2+i\epsilon} \; *$$

let $\langle f|S|i\rangle = 1 + i(2\pi)^4\delta(\text{mom})\langle f|T|i\rangle$

then

$$T_{fi} = -i\lambda + (-i\lambda)^2 i\left(V(s)+V(t)+V(u)\right)$$

where $* = (-i\lambda)^2 i\, V(s)$

Yukawa Theory $\mathcal{L} = \mathcal{L}_{K\cdot G.} + \mathcal{L}_{\text{Dirac}} + (-g\bar{\Psi}\Psi\phi)$ and leading order

contribution is 2nd : $\frac{1}{2!}(-ig)^2\int\langle p_3, p_4|T(\bar{\Psi}_\alpha(x)\Psi_\alpha(x)\phi(x)\bar{\Psi}_\beta(y)\Psi_\beta(y)\phi(y))|p_1,p_2\rangle d^4x\, d^4y$

creation — !? — contain anihil. ops

have suppressed spin labels! ie $|p_1s_1\, p_2s_2\rangle$

fermion -fermion elastic scat.

scalar

fermion-antifermion elastic scattering

Wicks theorem works just the same for fermions but with sign changes coz anticom.... but same end result.

The contraction $\Psi(x)|p,s\rangle$ gives: $\hookrightarrow e^{-ip\cdot x}\; u_s(p)$

this conserves mom while this just stays.

$\bar{\Psi}$ can contract with final state fermions. Ψ can contract with final state antifermions and $\bar{\Psi}$ can contract with initial state antifermions, giving $\bar{V}_s(p)$.

There are no sym. factors here, coz - signs....

prop: $\frac{i}{p^2-m^2+i\epsilon}$ as in ϕ^4.

fermion prop : $\frac{i(\gamma\cdot p + \text{Mass})_{\alpha\beta}}{(p^2-M^2+i\epsilon)}$

(contract with external spinor)

must sum over spinor indices. Get trace on closed loop.

Overall sign dep on order of multifermion state. Fermion number (charge...) is conserved.

So we have the rules : (mom space)

(1) Propagators as above

(2) Vertices $\succ$- - - give $(-ig)$

(3) External legs : in scalar - - ->- -< 1

out scalar >- ->- - - 1

in fermion →< $u_s(p)$

out fermion >→ $\bar{u}_s(p)$

in a/ferm >⇄ $v_s(k)$

out a/ferm ⇄< $\bar{v}_s(k)$

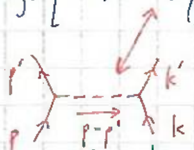
(4) Get the sign right

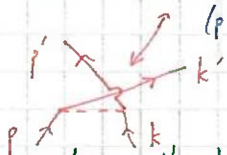
(5) Integrate, cons. momentum etc.

Eg's 2 Fermion scattering

$$iT = (-ig)^2\left[\bar{u}(p')u(p)\frac{1}{(p'-p)^2-m^2}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{1}{(p'-k)^2-m^2}\bar{u}(k')u(p)\right]$$

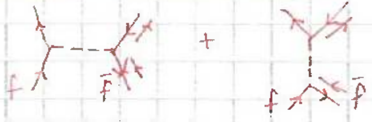
off shell so don't need $+i\epsilon$





(may wish to sum over spin indices - they have been suppressed)

Fermion - Antifermion Scattering



Classical Maxwell Theory

have gauge pot. $a^\mu = (a_0, \underline{a})$
$\Rightarrow a_\mu = (a_0, -\underline{a})$

Then can construct e/m field tensor:

$$\boxed{F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu}$$

then have: electric field $E_i = F_{0i} = \partial_0 \quad - \partial_i$ $\rightarrow \underline{E} = -\underline{\nabla}a_0 - \dot{\underline{a}}$

and magnetic field $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$ $\rightarrow \underline{B} = \underline{\nabla}\times\underline{a}$

Construct Lagrangian density:

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \underbrace{\frac{1}{2}|\underline{E}|^2}_{\text{Kinetic Part}} - \underbrace{\frac{1}{2}|\underline{B}|^2}_{\text{Potential Part}}$$

ie $\underline{E}$ represents dynamics of e/m.

→ the variational eq'n is $\boxed{\partial_\mu F^{\mu\nu} = 0}$ ie. the dynamical Maxwell eq'ns - those that are changed by presence of matter, div $\underline{E} = 0$ and $\dot{\underline{E}} =$ curl $\underline{B}$

Def'n of $F_{\mu\nu} \Rightarrow$ Bianchi Identities :

ie $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

these are the other ½ of the Maxwell eq'ns, $\underline{\nabla}\cdot\underline{B} = 0$ and $\dot{\underline{B}} = -$ curl $\underline{E}$.

Now, there is no dep on $\dot{a}_0$ in $\mathcal{L}_{em}$ so a_0 is not really dyn. var : $\underbrace{\nabla^2 a_0 = -\underline{\nabla}\cdot\dot{\underline{a}}}_{\text{Poission, source}} \Rightarrow a_0(\underline{x}) = \int d^3\underline{x}' \frac{\underline{\nabla}\cdot\dot{\underline{a}}\; x'}{4\pi|\underline{x}-\underline{x}'|}$

Quantised Maxwell Theory

Now use A for gauge potential, not a.

Impose equal time commutation relations $[A_i(\underline{x}), A_j(\underline{x}')] = 0$

$$[A_i(\underline{x}), \pi_j(\underline{x}')] = i\left(\delta_{ij} - \nabla^{-2}\partial_i\partial_j\right)\delta^{(3)}(\underline{x}-\underline{x}') \quad [\pi_i(\underline{x}), \pi_j(\underline{x}')] = 0$$

this ensures that $\underline{\nabla}\cdot\underline{A}$ and $\underline{\nabla}\cdot\underline{\pi}$ have trivial commutators with everything else - only really has meaning when insert integral...

Work in C.G.

$\underline{\nabla}\cdot\underline{a} = 0 \Rightarrow$ source $= 0 \Rightarrow a_0 = 0$

So $\underline{E} = -\dot{\underline{a}}$ and get $\boxed{\ddot{\underline{a}} = \nabla^2\underline{a}}$ ie $\boxed{\partial^2\underline{a} = 0}$

Wave solutions:

$\underline{a} = \underline{e}\exp(-ik\cdot x)$ $\underbrace{k^2 = 0}_{c=1}$ and $\underline{k}\cdot\underline{e} = 0$

Canonical mom. conjugate to $\underline{a}$ is $\underline{\pi} = -\underline{E}$ $(= \dot{\underline{a}})$

$\underline{\nabla}\cdot\underline{\pi} = 0$ in C.G. so $\exists$ only 2 indep components.

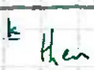
Mode expansions are :

$$\underline{A}(\underline{x}) = \int\frac{d^3\underline{k}}{(2\pi)^3}\frac{1}{\sqrt{2|\underline{k}|}}\left(\underline{a}_{\underline{k}}e^{i\underline{k}\cdot\underline{x}} + \underline{a}_{\underline{k}}^\dagger e^{-i\underline{k}\cdot\underline{x}}\right)$$

$$\underline{\pi}(\underline{x}) = \int\frac{d^3\underline{k}}{(2\pi)^3}(-i)\sqrt{\frac{|\underline{k}|}{2}}\left(\underline{a}_{\underline{k}}e^{i\underline{k}\cdot\underline{x}} - \underline{a}_{\underline{k}}^\dagger e^{-i\underline{k}\cdot\underline{x}}\right)$$

Require $\underline{k}\cdot\underline{a}_{\underline{k}} = 0$ and $\underline{k}\cdot\underline{a}_{\underline{k}}^\dagger = 0$ so only have 2 operators for each $\underline{k}$ not $\underline{3}$ - component // to $\underline{k}$ is absent.

For each $\underline{k}$, introduce polarisation vectors : $\underline{e}_1$



then write:

$$\underline{a}_{\underline{k}} = a_{\underline{k},1}\underline{e}_1 + a_{\underline{k},2}\underline{e}_2$$

$$\underline{a}_{\underline{k}}^\dagger = \underbrace{a_{\underline{k},1}^\dagger\underline{e}_1 + a_{\underline{k},2}^\dagger\underline{e}_2}$$

these destroy and create photons of momentum $\underline{k}$, energy $|\underline{k}|$ and electric polarisation along $\underline{e}_i$

If refer to sph. basis set, get circular polarisation.

The commutation relations become $[\underline{a}_{\underline{k}}, \underline{a}_{\underline{l}}] = 0$

$[\underline{a}_{\underline{k}}^\dagger, \underline{a}_{\underline{l}}^\dagger] = 0$

$$[a_{\underline{k},i}, a_{\underline{l},j}^\dagger] = (2\pi)^3\delta^{(3)}(\underline{k}-\underline{l}) \times \left(\delta_{ij} - \frac{k^i k^j}{|\underline{k}|}\right)$$

So $k^i a_{\underline{k}}^i = 0$ enforcing C.G.

Q.F.T. ⑨

QED - Propagators

We have e/m part of QED prop.
The H.P. e/m field is

$$\underline{A}(x) = \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{\sqrt{2|\underline{k}|}} \left(\underline{a}_{\underline{k}} e^{-ik\cdot x} + \underline{a}_{\underline{k}}^{\dagger} e^{ik\cdot x} \right)$$

then, in usual way, get $\langle 0 | T A_i(x) A_j(y) | 0 \rangle$

$$= \boxed{D^{tr}_{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2-i\varepsilon)} \left(\delta_{ij} - \frac{k_i k_j}{|\underline{k}|^2} \right) e^{-ik\cdot(x-y)}}$$

To get time part $D_{00}(x-y)$ must look at QED $\mathcal{L}$:

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

$$= \mathcal{L}_{em} + \mathcal{L}_{Dirac} + -e \underbrace{\bar{\Psi}\gamma^\mu\Psi}_{j^\mu} A_\mu$$

and $D_\mu = \partial_\mu + ieA_\mu$

Field eq'ns are $\partial_\mu F^{\mu\nu} = e\underbrace{\bar{\Psi}\gamma^\nu\Psi}$

So ej^ν electric current $\longrightarrow j^\nu$

and $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ w.r.t. $\bar{\Psi}$.

Now in C.G. $\underline{\nabla}\cdot\underline{A} = 0$
So zero bit of is $\underbrace{\partial_i F^{i0}}_{\text{div } E} = \underbrace{ej^0}_{= e\rho}$

$\Rightarrow -\nabla^2 A_0 = e\bar{\Psi}\gamma^0\Psi$
$= e\underbrace{\Psi^\dagger\Psi}$

$\Rightarrow -\nabla^2 A_0 = e\rho$

this has solution:

$$A_0(\underline{x}) = e\int \frac{\rho(\underline{x}')}{4\pi|\underline{x}-\underline{x}'|} d^3\underline{x}'$$

The Q.E.D. Hamiltonian is: $H =$

$$\int\left(\frac{1}{2}\underline{\Pi}\cdot\underline{\Pi} + \frac{1}{2}\underline{B}\cdot\underline{B} - i\bar{\Psi}\underline{\gamma}\cdot\underline{\nabla}\Psi + m\bar{\Psi}\Psi\right)d^3\underline{x}$$

$$\underbrace{+\frac{e^2}{2}\int\frac{\rho(\underline{x})\rho(\underline{x}')\,d^3\underline{x}\,d^3\underline{x}'}{4\pi|\underline{x}-\underline{x}'|}}_{\text{instantaneous Coulomb interaction}} \underbrace{-e\int \underline{j}\cdot\underline{A}\, d^3\underline{x}}_{\text{photon exchange}}$$

instantaneous Coulomb interaction
Prop is $D^{Coul}_{00}(x-y)$
$= \frac{1}{4\pi|\underline{x}-\underline{y}|}\delta(x^0-y^0)$

photon exchange
Prop. is $D^{Coul}_{ij}(x-y)$ which is the transverse photon propagator $D^{tr}_{ij}(x-y)$

So total propagator is $D^{Coul}_{\mu\nu}(x-y)$
In momentum space,

$$D^{Coul}_{ij}(k) = \frac{i}{\text{etc...}} \qquad D^{Coul}_{00} = \frac{i}{|\underline{k}|^2}$$

Now, $D^{Coul}_{\mu\nu}$ always appears as $c_\mu D^{Coul}_{\mu\nu} d_\nu = i\left(\frac{\underline{c}\cdot\underline{d}}{k^2} - \frac{k_0^2 c_0 d_0}{k^2|\underline{k}|^2} + \frac{c_0 d_0}{|\underline{k}|^2}\right)$

Current conservation,
i.e. $k_\mu c^\mu = 0 = k_0 c^0 - \underline{k}\cdot\underline{c}$
allows us to get this term

$$= \frac{i}{k^2}(\underline{c}\cdot\underline{d} - c_0 d_0) = \frac{i}{k^2}c^\mu d_\mu$$

[Gauge Inv. allows us to write
$D_{\mu\nu}(k) = \frac{-i}{k^2}\left(g_{\mu\nu} + \lambda\frac{k_\mu k_\nu}{k^2}\right)$
$\lambda = 0$ - Feynman gauge prop.
$\lambda = -1$ - Landau " "]

$$\Rightarrow \boxed{D_{\mu\nu}(k) = \frac{-i}{k^2}g_{\mu\nu}}$$

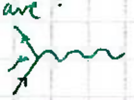
due to Current conservation

Lorentz Invariant Propagator.

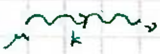
Feynman Rules for Q.E.D.

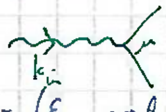
We have fermion lines, ⟶ and photon lines ⇝.
Fermion rules are as in Yukawa theory
(ie have u_s, $\bar{u}_s$, $v_s(p)$, $\bar{v}_s(p)$, fermion propagator sum over spin labels and spinor indices)

Now we have:

(i) Vertex  get $-ie\gamma^\mu$

Contract.

(ii) Photon propagator  get $\frac{-i}{k^2+i\varepsilon}g_{\mu\nu}$

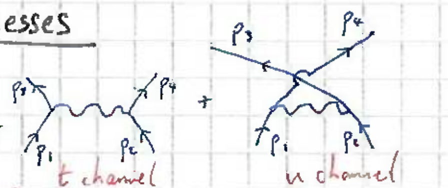
(iii) External photon lines  get ε_μ ← Pol vector

where $\varepsilon_\mu = (\varepsilon_0, \cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2)$
$\varepsilon_0 = 0$ in C.G.

$\underline{\varepsilon}\cdot\underline{a}^\dagger_{\underline{k}}$ creates a photon with polarisation $\underline{\varepsilon}$.

Examples of Q.E.D. Processes

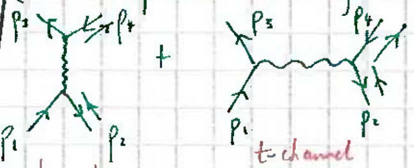
(i) electron scattering $e^-e^- \to e^-e^-$



t channel
Prop $\propto \frac{1}{t}$

u channel
Prop $\propto \frac{1}{u}$

(ii) electron positron scattering $e^+e^- \to e^+e^-$

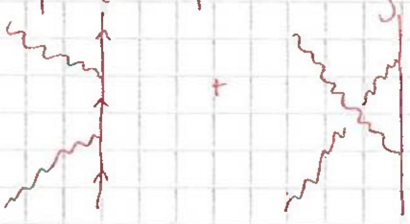


s-channel

t-channel

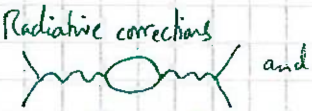
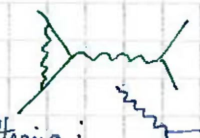
All these tree diags have no integrations
- they are all $O(e^2)$
- strength is $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$

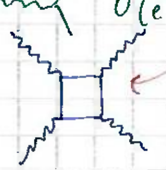
(iii) Compton (electron photon) Scattering $e\gamma \to e\gamma$:



(iv) $e^+e^- \to 2\gamma$

e^+e^- can form a bound state called positronium - decays → 2γ.

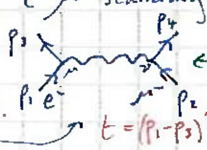
∃ Radiative corrections
eg  and  $O(e^4)$

Can get Photon-Photon Scattering: 

Divergences vanish via current conservation i.e. Gauge Invariance.

Muons - same as Q.E.D. except $m_\mu \simeq 200\, m_e$. Within Q.E.D. μ^- stable but actually decay → e^-, ν_e via Weak interaction, $W^\pm$ bosons.

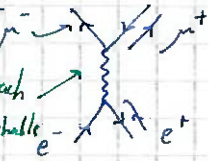
eg $e^-\mu^-$ scattering



$t = (p_1-p_3)^2$

← only one diag for each → as e, μ distinguishable

$e^+e^- \to \mu^+\mu^-$



Non-relativistic limit of $e^-\mu^-$ scattering

Amplitude:

$$i\mathcal{T} = -e^2 \underbrace{\bar{u}_{s_3}(p_3)\gamma^\mu u_{s_1}(p_1)}\frac{(-ig_{\mu\nu})}{(p_3-p_1)^2}\underbrace{\bar{u}_{s_4}(p_4)\gamma^\nu u_{s_2}(p_2)}_{\sim 2m},$$

For small $|\underline{q}|$ only get contrib. from $s_1 = s_3 = s$ and $s_2 = s_4 = s'$
γ^0 dominates and $\sim 2m$

So $i\mathcal{T} \simeq \frac{-e^2(2m)^2 i}{|\underline{q}|^2}$

Now work out scattering amplitude in Born approx, using a potential - find that pot must $= \frac{e^2}{4\pi|\underline{r}|}$ because in Born ap.

$\text{F.T.}\left(\frac{e^2}{4\pi r}\right) = \frac{e^2}{|\underline{q}|^2}$. +ve ∴ repulsive

If do for $e^-\mu^+$, get −ve sign from ordering → attractive.

Q.F.T. ⑩

Non-Abelian Gauge Theory

In QED there is G Inv.:
under $\psi(x) \to e^{ie\chi(x)}\psi(x)$
and $a_\mu \to a_\mu - \partial_\mu \chi(x)$

Hence gauge covariant derivative:
$$\underbrace{D_\mu\psi \to e^{ie\chi} D_\mu \psi}_{\text{transforms like } \psi.}$$

Yang + Mills extended this:

let $\Phi(x) \longrightarrow g(x)\,\Phi(x)$

Φ: n component multiplet of scalar or dirac fields

$g(x)$: some matrix $g \in G$, any compact Lie group of n×n matrices.

Now let $D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi$

require $D_\mu \Phi \to g D_\mu \Phi$

$$\Rightarrow \boxed{A_\mu \to g A_\mu g^{-1} - (\partial_\mu g) g^{-1}}$$

A_μ is an n×n matrix for each μ and

$\boxed{A_\mu \in \text{Lie}(G)}$! If $G = SU(n)$ get traceless antiherm.
eg QED $A_\mu = iea_\mu$!

Now let $\Psi(x)$ be a matrix (n×n) of fields (not vector) and let $\Psi(x) \in \text{Lie}(G)$. Gauge trm: $\boxed{\Psi \to g\Psi g^{-1}}$
then the adjoint covariant derivative, $D_\mu \Psi = \partial_\mu \Psi + [A_\mu, \Psi]$ transforms like Ψ:

$$D_\mu \Psi \to g(D_\mu \Psi) g^{-1}$$

In fact, there is a different covariant derivative for each rep of G. This was the fundamental rep. This was the adjoint rep.

Field Tensors

The field is a curvature - derive via the commutator of covariant derivatives.

Abelian Case (- QED)
Covariant deriv is
$D_\mu \phi = \partial_\mu \phi + iea_\mu \phi$
$\Rightarrow [D_\mu, D_\nu]\phi = ief_{\mu\nu}\phi$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

Non Abelian Case
Cov. deriv is $D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi$
now we get
non linear
$$[D_\mu, D_\nu]\Phi = \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])}_{\text{the Y-M field tensor, } F_{\mu\nu}.}\Phi$$

$F_{\mu\nu}$ is in Lie(G) coz
- have n×n matrix for each μ, ν.

We can define the Y-M elec. and Y-M mag fields, $E_i = F_{0i}$ and $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$ but they are matrices.

Under a gauge trm,
$$\boxed{F_{\mu\nu} \longrightarrow g F_{\mu\nu} g^{-1}}$$

Unlike e/m where $f_{\mu\nu} \to f_{\mu\nu}$
$\Rightarrow E_i\ B_i$ not observables.

We can introduce a coupling constant
ie let $D_\mu \Phi = \partial_\mu \Phi + eA_\mu \Phi$

Then $F_{\mu\nu} = \underbrace{\partial_{[\mu}A_{\nu]}}_{\text{gluons come from here}} + \underbrace{e[A_\mu, A_\nu]}_{\text{now this bit is small}}$

e is important in P.T. of quantised Y.M. theories.

Can introduce basis set of matrices for Lie(G),
$\{T^a\}$ for $1 \leq a \leq \dim(\text{Lie}(G))$
(orthonormal if $\text{Tr}(T^aT^b) = \kappa\delta^{ab}$)

the Lie algebra is closed under commutation:
$[T^a, T^b] = f^{abc}T^c$ Then:
$$A_\mu = \underbrace{A_\mu^a}_{\text{the components of } A_\mu} T^a$$
(eg SU(2), $f^{abc} = \varepsilon^{abc}$)

Then $D_\mu \Phi = \partial_\mu \Phi + A_\mu^a T^a \Phi$ w.r.t. Lie(G)

$$\Rightarrow F_{\mu\nu} = \underbrace{(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{bca}A_\mu^b A_\nu^c)}_{F_{\mu\nu}^a} T_a$$

Bianchi (Jacobi) applies here too.

Yang-Mills Lagrangians

require to be a gauge invariant Lorentz Scalar ...

Can construct various objects like this eg:

(1) $\Phi^\dagger \Phi$ For $G = U(n)$

2) $(D_\mu \Phi)^\dagger D^\mu \Phi$
$\Rightarrow$ scalar.

(3) For Dirac fields
$\bar{\Psi}(i\gamma^\mu D_\mu \Psi - m\Psi)\Psi$

(4) $\text{Tr}(\Psi^2)$ for adjoint Ψ
[NB $\text{Tr}\,\Psi = 0$ coz $\in$ Lie(G)]

(5) $\text{Tr}(F_{\mu\nu}F^{\mu\nu})$
this is $\propto$ Y-M Lagrangian.

Example Q.C.D. Gauge group is SU(3). Each quark field is a 3 (colour) component field: q.

$$\mathcal{L} = \text{const.}\,\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_{\text{6 quark types}} \bar{q}(i\gamma^\mu D_\mu - m_q)q$$

each type, u d c s t b has different mass

The Classical Field Equations

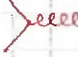
Pure Y-M theory (ie. no quarks)
gives: $D_\mu F^{\mu\nu} = 0$ ie. $\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$

With matter ie. Dirac fields, get

$$D_\mu F^{\mu\nu} = j^\nu \text{ where } j^\nu \sim \bar{q}\, i\gamma^\nu q.$$

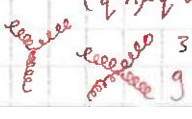
In components, $(D_\mu F^{\mu\nu})^a = \bar{q}\, i\gamma^\nu T^a q$

also have $(i\gamma^\mu D_\mu - m_q)q = 0$
ie Dirac eq'n for quarks.



q-g vertex ($\bar{q}A_\mu q$ in)

Feynman rules involve gluon propagator
quark propagator



3 and 4 g vertices.

Renormalisation.

Parameters in the models aren't the ones we measure eg QED - we found pot is $\frac{e^2}{4\pi r}$

but $\frac{(e_{\text{actual}})^2}{4\pi r} = \frac{e^2 + O(e^4) + \cdots\cdots}{4\pi r}$ so, reexpress e as $e(e_R)$

e_R

In scalar ϕ^4 theory $T = -\lambda - \lambda^2\underbrace{(V(s) + V(t) + V(u))}_{\text{divergent integrals.}}$

for

to 2nd order

Use the physical quantity: $-T\underbrace{(s = 4m^2, t = 0, u = 0)}_{\text{at rest, no scat, - arb. renorm point.}} = \lambda_R$

$\Rightarrow \lambda_R = \lambda + \lambda^2(V(4m^2) + V(0) + V(0))$

$\Rightarrow \lambda = \lambda_R - \lambda_R^2(V(4m^2) + 2V_0) + O(\lambda_R^3)$

$\Rightarrow T = -\lambda_R - \lambda_R^2[V(s) - V(4m^2) + V(t) - V(0) + V(u) - V(0)]$

these differences fall off rapidly at high loop momenta $\therefore$ T remains finite - equiv. of mom cut-off.

TPS ②

WKB Method

Obtains assymptotic solution of $\psi'' + k^2(x)\,\psi = 0$

if S.E. then $k^2 = \frac{2m}{\hbar^2}(E-V)$

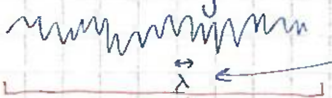
let $\psi = A(x)\,e^{iS(x)}$ as a first approx : get from Im. part,

get $\frac{A'}{A} = -\frac{S''}{2S'}$

So $A = \frac{a}{\sqrt{S'}}$. Real part gives:

$$(S')^2 = k^2 + \frac{A''}{A}$$

Now k^2 ie. $V(x)$, the potential varies on a length scale λ



let l be the size of region of interest - ie gradient of $A \simeq \frac{A}{l}$ (scale of variation of A)

now $k^2 \sim \frac{1}{\lambda^2}$

and $\frac{A''}{A} \sim \frac{1}{l^2}$

So $\frac{1}{\lambda^2} + \frac{1}{l^2} \simeq \frac{1}{\lambda^2}$

if $\underline{l \gg \lambda}$

So $S'^2 \simeq k^2$

and $\psi(x) = \frac{a_1}{\sqrt{k}} \exp\left(i\int_{x_0}^{x} k\,dx\right) + \frac{a_2}{\sqrt{k}} \exp\left(i\int_{x_0}^{x} k\,dx\right)$

for real k.

so if k is indep. of x, get plane wave solutions.

Assymptotic Series

let $S_n = \sum_{k=0}^{n} \frac{a_k}{x^k}$

then S_n is an assymptotic representation of $f(x)$ if:

$$\boxed{\lim_{x\to\infty} x^n\left(S_n(x) - f(x)\right) = 0}$$

ie the difference $\longrightarrow 0$ faster than $x^n \longrightarrow \infty$.

now $\underline{S_0 = a_0 = f(\infty)}$

- can find a_1 from def'n:

$x(S_1 - f) = 0$

$\therefore x\left(a_0 + \frac{a_1}{x} - f\right) = 0$

$\therefore a_1 = \lim_{x\to\infty} \left(f(x) - f(\infty)\right)$

Method of Steepest Descents

Consider the integral $J(x) = \int_C e^{xf(t)}\,dt$

let $t = t_0$ be a point where $f(t)$ is maximum.

no no no
no no
no no no

$J(x) = \int e^{xi\,\mathrm{Im}\,f(t)} \cdot e^{x\,\mathrm{Re}\,f(t)}\,d$

now - expand $f(t) \simeq f(t_0) + (t-t_0)\underset{0}{f'(t_0)} + \frac{(t-t_0)^2}{2} f''($

and let $(t-t_0) = \delta e^{i\alpha}$

then $J(x) = \int_{C\ (\text{phase } \alpha)} e^{xf(t_0)}\, e^{\frac{xf''(t_0)}{2}\delta^2 e^{2i\alpha}}\, d\delta\, e^{i\alpha}$

- want integral to be Gaussian so $xf''(t_0)e^{2i\alpha}$ must be $-$ve real $\Rightarrow$ determines phase, α.

Groups

Orthogonality of irrep matrices : $\sum_{\text{elements}} \Gamma^{\mu}_{ab}(u)\,\Gamma^{\nu}_{cd}(u^{-1}) = \underline{[G]}\,\delta_{\mu\nu}\delta_{ac}\delta_{bd}$

$\Rightarrow$ for characters, $\sum_{\text{elements}} \chi_\mu(u)\chi_\nu(u^{-1}) = [G]\,\Gamma^{\mu}_{aa}(I) \times \delta_{\mu\nu}$

or, if N_C is the number of elements in class C then

$$\sum_{\text{classes}} N_C\, \chi_\mu(C)\, \chi_\nu(C^{-1}) = [G]\,\delta_{\mu\nu}$$

now a character table is square because no. of irreps = no. of classes, so can run other way :

$$\sum_{\text{irreps}} N_C\, \chi_\mu(C)\, \chi_\mu(C'^{-1}) = [G]\,\delta_{CC'}$$

the summations also work over group elements.

Let $\underset{\text{rep}}{\Gamma_r} = \sum_\alpha a_\alpha \underset{\text{irreps}}{\Gamma_\alpha} \Rightarrow \chi_r = \sum_\alpha a_\alpha \chi_\alpha$

$$\Rightarrow \sum_{\text{elements}} \chi_r(u)\chi_\beta(u^{-1}) = \sum_{\text{elements}} \sum_\alpha a_\alpha \chi_\alpha(u)\chi_\beta(u^{-1})$$

$$= \sum_\alpha a_\alpha [G]\,\delta_{\alpha\beta}$$

So number of times irrep β occurs, a_β is :

$$\boxed{a_\beta = \frac{1}{[G]} \sum_{\text{elements}} \chi_r(u)\chi_\beta(u^{-1})}$$

if $\Gamma = \Gamma_1 \otimes \Gamma_2$ then

$$a_\beta = \frac{1}{[G]} \sum_{\text{elements}} \chi_1(u)\chi_2(u)\chi_\beta(u^{-1})$$

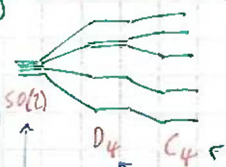
a Clebsch Gordan Coeff

TP3 ③

Application of Groups

QM - If have hamiltonian which is invariant under action of elements of a group then that group, G, is the sym. group for the system. ∃ an irrep for each level. The dimension of each irrep gives the degeneracy of the level. If perturb the system, so hamiltonian → a new one, then if sym of new one is less than old one, new symetry group is that of (G') Express old irrep (now rep) in direct sum of of new irreps. Dim. of new irreps gives new degeneracy.

eg. H atom (SO(2)) goes into D_4 crystal, then apply mag. field → C_4



$$\Gamma_{L=2} = \Gamma_0 \oplus \underset{(2D)}{\Gamma_\perp} \oplus \Gamma_{q^+} \oplus \Gamma_{q^-}$$

$$= \Gamma_0 \oplus 2\Gamma_{q^+} \oplus \Gamma_+ \oplus \Gamma_-$$

2 levels with quadrupole type sym. - not nec. degenerate!

NORMAL MODES OF SHAPES

- a shape - eg molecule has a certain sym. group. The vector space of position coordinates of atoms will induce a representation of G. If then can block diagonalise this as a Direct sum of irreps of G, then have found the degeneracy.

called Γ_T. now $\chi_T(C\phi) = (N_{axis} - 2)(1 + 2\cos\phi)$ for rotation of ϕ. Naxis is the number of atoms on the axis of rotation.

For rotation and reflection (about z axis and in x-y plane), $\chi_T = -1 + 2\cos\phi$.

FREE ENGERY INVARIANTS

Free energy is invariant under any operation of a crystal's symmetry group. eg the piezoelectric effect: $F \propto \gamma_{ijk} P_i \sigma_{jk}$

γ_{ijk}: Piezo tensor; P_i: Polarisation vector; σ_{jk}: stress tensor (sym)

rep $V \otimes [V^2]$

This rep will decompose into irreps of the sym. group. The number of trivial irreps needed gives the no. of indep. components in γ_{ijk}.

Sym square: $\chi_{[V^2]} = \frac{1}{2}\left[\chi^2(u) \pm \chi(u^2)\right]$

anti sym...

Stochastic Physics - Basics

An OUTCOME : $\underline{x} = (x_1 \cdots x_n)$ eg $\underline{x} = (x, \dot{x})$ for particle in one dimension or $\underline{x} = (x\ y\ z)$ for random walk.

A joint probability distribution function f is s.t.

$f(\underline{x})\, d^n\underline{x}$ is prob of $\underline{x}$ being in $d^n\underline{x}$

An average of $y(\underline{x})$ is :

$$\overline{y(\underline{x})} = \int_{\text{all space}} y(\underline{x}) f(\underline{x})\, d^n\underline{x}$$

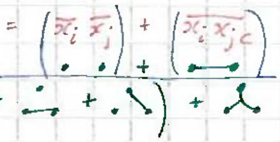
Moments are averages of products (of random variables)

eg $\overline{x} \equiv$ mean, $\overline{x_i x_j} \equiv$ correlation Fn.

$\overline{x_i x_j} - \overline{x_i}\,\overline{x_j} \equiv$ covariance

CUMULANTS: $\overline{x_i x_j} = \left(\overline{x_i}\ \overline{x_j}\right) + \left(\overline{x_i x_j}_c\right)$

$\overline{x_i x_j x_k} =$



Cumulants are last terms eg

Something v. important:

$$\boxed{f_k = \sum_{m=0}^{\infty} \frac{\overline{(-ikx)^m}}{m!} = \exp\left(\sum_{m=1}^{\infty} \frac{\overline{(-ikx)^m_c}}{m!}\right)}$$

Because $f_k = \overline{\exp(-ikx)}$ for any $f(x)$.

So $\ln f_k = \sum_{m=1}^{\infty} \frac{\overline{(-ikx)^m_c}}{m!}$

So $\boxed{\overline{x^m} = i^m \frac{d^m}{dk^m} f_k \Big|_{k=0}}$ and $\boxed{\overline{x^m}_c = i^m \frac{d^m}{dk^m} (\ln f_k)\Big|_{k=0}}$

for a Gaussian P.D.F., $f_k = e^{-\frac{k^2\sigma^2}{2}}$

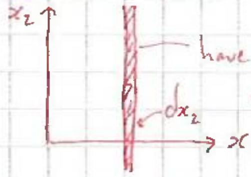
so $\boxed{\overline{x^{2m}} = \frac{(2m)!}{2^m m!}\sigma^{2m}}$

for Gaussian P.D.F. cumulants vanish beyond second order.

Dependence vs. Correlation

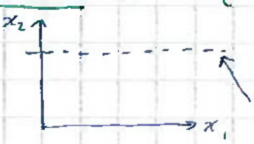
Unconditional P.D.F. : $f(x_1) = \int f(x_1, x_2)\, dx_2$

ie



have integrated out x_2. ie. add over all possible values of x_2.

Conditional PDF : $f(x_1 | x_2) = \frac{f(x_1, x_2)}{\int f(x_1, x_2)\, dx_1}$



ie constrain x_2 to be a certain value - but must normalise

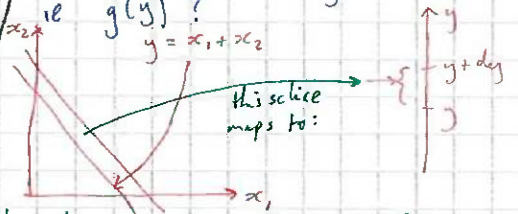
n random variables are INDEPENDENT if can write:

$$\boxed{f(\underline{x}) = f_1(x_1) f_2(x_2) \cdots\cdots f_n(x_n)}$$

Independence ⟺ all cumulants vanish.

Independent Random Variables

If have $f(\underline{x})$, and $y = x_1 + x_2 + \cdots x_n$ then what is P.D.F. for y ie $g(y)$?



$g(y)\,dy$ = prob of getting x_1 and x_2 such that $y = x_1 + x_2$

$$= \int \underbrace{f(x_1, x_2)\, dx_1 dx_2}_{\text{prob of getting } x_1 \text{ and } x_2.} \text{ s.t. } y = x_1 + x_2$$

$$g(y) = \int f(x_1, y - x_1)\, dx_1$$

$$= \int f_1(x_1) f_2(y - x_1)\, dx_1$$

if INDEPENDENT !!

So by the convultion theorem:

$g_k = f_{1k} f_{2k}$

$\therefore g_k = f_{1k} f_{2k} f_{3k}$

ie $\prod_{1}^{n} f_{nk}$

CENTRAL LIMIT THEOREM:

g → Gaussian as n → ∞.

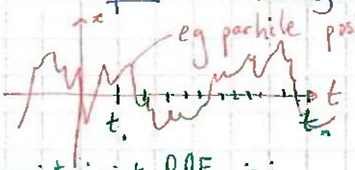
eg take $f_1 = f_2$ etc then $g_k = (f_k)^n$

$$g_k = \Big(1 - ik\underset{\text{zero}}{\cancel{\overline{x}}} - \frac{k^2 \overset{\sigma^2}{\overline{x^2}}}{2} + \cdots\Big)^n = \left(1 - \frac{k^2 n\sigma^2}{2n}\right)^n \to e^{-\frac{k^2\sigma^2 n}{2}} \text{ as } n \to \infty$$

ie Gaussian, variance = $n\sigma^2$

Stochastic Processes

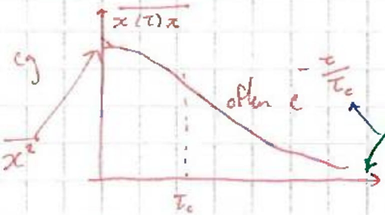
This is an outcome regarded as a path or a history



n-point joint P.D.F. is:

$f_n(x_1, t_1; x_2, t_2; \dots x(t_n))$

$= f(x(t))$ sampled at n times



Integrate out n−2 variables to get the 2 point P.D.F. ie correlation function to lowest order.

$f_2(x(t_1); x(t_2))$

x_1 x_2

Then have $\overline{x_1 x_2}$

Stationary Process

means indep. of time origin - so can choose $t_1 = t$ w.l.o.g.

$t_2 = \tau + t$ then

$\overline{x_{t+\tau} x_t} \rightarrow \overline{x(\tau)x}$

(also choose $\overline{x(t)} = 0$)

no correlation ⇒ $\overline{x^2} = \bar{x}^2 = 0$

Deterministic Process have full memory eg SHM ....

Markov Process: if know present, future is indep of past.

eg Random Walk. Then $\tau_c \longrightarrow 0$.

Wiener Process (1-D Random Walk)

sum of $n \left(= \frac{t}{\Delta t}\right)$ indepent steps = total distance.

ie $v_i = \frac{l_i}{\Delta t} \rightarrow x(t) = l_1 + l_2 \cdots l_n$ $\left(\text{i.e. } \int_0^t v(t')dt' = x\right)$

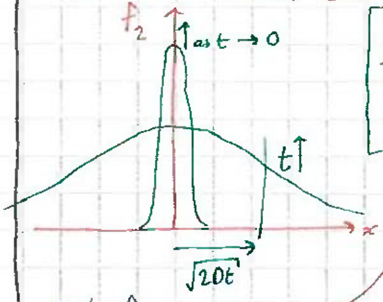
Assume $\overline{l_i} = 0$ (ie + or − equally likely)

and that $\overline{l_i^2} = 2D\Delta t$. Then because Σ var = var Σ

$\overline{x^2(t)} = 2Dt$. Central limit theorem ⇒

$$\boxed{f_2(x, t | 0, 0) = \frac{e^{\frac{-x^2}{2(2Dt)}}}{\sqrt{4\pi Dt}}}$$

2 pt. joint PDF condition on $x(0) = 0$



Call f_2 $G(x, t; 0, 0)$

it is the G.F. for:

$$\boxed{\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) \leftarrow \text{Diffusion operator, } L_0}$$

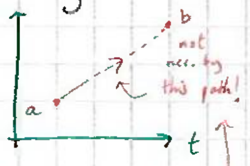
retarded → $t > 0$ $L_0 G = 0$

$t = 0$ $L_0 G = \delta$-f'n

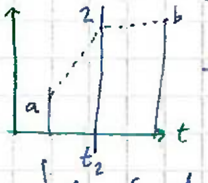
F.T. → $G_k(t; 0, 0) = e^{-Dk^2 t}$

So Dk^2 is relaxation (decay) rate of the kth frequency in $G(x, t; 0, 0)$.

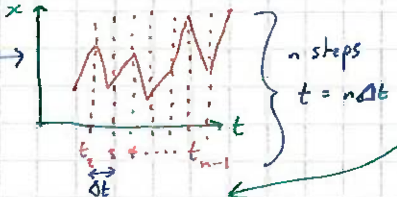
Writing a Green F'n as Path Integral



$G(x_b, t_b; x_a, t_a)$ is prob of going from a → b



$G_{ab} = \int G_{a2} G_{2b} \, dx_2$



$G_{ab} = \int G_{a2} \cdots G_{n-1\,b} \, dx_2 \cdots dx_{n-1}$

$G_{ab} = \int e^{-\sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)^2}{4D\Delta t}} \underbrace{\frac{dx_2 \cdots dx_{n-1}}{(4\pi D\Delta t)^{\frac{n-1}{2}}}}$ → $\mathcal{D}x(t)$

take lim $n \to \infty$ and $\Delta t \to 0$

$\Rightarrow G_{ab} = \int e^{\frac{1}{4D}\int_a^b \dot{x}^2 dt} \mathcal{D}x(t)$

$\Rightarrow G_{ab} = \int e^{-A[x(t)]} \mathcal{D}x(t)$

where $A[x(t)] = \int_a^b L(x, \dot{x}) \, dt$

$\mathcal{N}e^{-A}$ looks like prob. of "path" so

$\overline{x(t)} = \int x(t) e^{-A} \mathcal{D}x(t)$

etc for higher moments of paths

Ginzburg - Landau Model

FOURIER REP: expand random path as $x(t) = \frac{1}{\sqrt{t_0}} \sum_\omega x_\omega e^{-i\omega t}$ in interval $0 \longrightarrow t_0$

where $\omega = \frac{2\pi n}{t_0}$ $-\infty \leq \omega \leq \infty$

$x_\omega = \frac{1}{\sqrt{t_0}} \int_0^{t_0} x(t) e^{i\omega t} dt$ from orthogonality,

WIENER-KHINCHINE THEOREM

$\overline{x^*_{\omega+\Omega} x_\omega} = \frac{1}{t_0} \int d\tau dt \, \overline{x(\tau)x} \, e^{-i(\omega+\Omega)(t+\tau)} e^{i\omega t}$

$= \int d\tau \, \overline{x(\tau)x} \, e^{-i\omega\tau} \cdot \delta_{\Omega,0}$

So all "cross terms" are zero

get: $\boxed{\overline{|x_\omega|^2} = \text{F.T. } \overline{x(\tau)x}}$

power spectrum is F.T. of auto corr. f'n

let $\tau = 0$, then

$\int \overline{|x_\omega|^2} \frac{d\omega}{2\pi} = \overline{x^2}$

$d\nu$

So can interpret $\overline{|x_\omega|^2}$ as mean square value per unit frequency range

Expand $A[x(t)]$ as even powers:

$A[x(t)] = \frac{1}{2}\int a(t, t') x(t) x(t') dt dt' + x^4 \text{ term} + \dots$

continuous quadratic form.

$= \left[\frac{a}{2} x^2\right] + \left[\frac{b}{4} x^4\right] + \dots$

for stationary process, → $a(t - t')$

now use to get

$A[x_\omega] = \sum_\omega \frac{a_\omega}{2} |x_\omega|^2 + \frac{b}{4t_0} \sum_{\substack{\omega_1\omega_2\omega_3\omega_4 \\ \text{s.t. sum}=0}} x_{\omega_1} x_{\omega_2} x_{\omega_3} x_{\omega_4} + \dots$

GAUSSIAN LIMIT ≡ $b = 0$ ie. no quadratic terms.

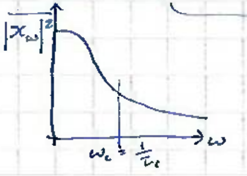
from here can read off variance $= \overline{|x_\omega|^2} = \frac{1}{a_\omega}$

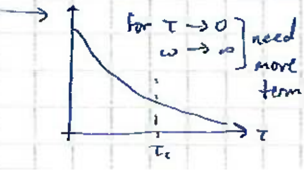
$= \frac{1}{a(1 + \omega^2\tau^2)}$

for $\tau_c = \sqrt{\frac{c}{a}}$

a Lorentzian

So $\overline{x(\tau)x} = \overline{x^2} e^{-|\tau|/\tau_c}$ ← ω-K





for τ → 0, ω → ∞ need more terms

now let $a_\omega = a + c\omega^2$ valid for low freq only!

and F.T. back to t,

Then $\boxed{A[x(t)] = \frac{1}{2}\int \left(ax^2 + \frac{b}{2}x^4 + c\dot{x}^2\right) dt}$

G-L Action.

TP3 ⑤

The Langevin Model

L-eq'n is eq'n for particle in liquid:

$$\kappa x + m\ddot{x} = f_{liq} + f_{ext}$$
$$= \eta - \gamma\dot{x} + f_{ext}.$$

↑ random, white power spectrum and zero mean

So:

$$\kappa x + \gamma\dot{x} + m\ddot{x} = f_{ext}$$

FT ⇒ $\overline{x_\omega} = \alpha_\omega f_\omega$ $\quad \alpha_\omega = \frac{1}{(\kappa - i\gamma\omega - m\omega^2)}$

this kills $\eta \Leftrightarrow \overline{\eta} = 0$

for $f_{ext} = 0$,

(indep of frequency)

$$x_\omega = \alpha_\omega \eta_\omega \Rightarrow \overline{|x_\omega|^2} = |\alpha_\omega|^2 \overline{|\eta_\omega|^2}$$

Equipartition gives $\frac{1}{2}\kappa\overline{x^2} = \frac{1}{2}kT$

now $\overline{x^2} = \int \overline{|x_\omega|^2} \frac{d\omega}{2\pi}$ (W-K)

So $$\overline{x^2} = \frac{\overline{|\eta_\omega|^2}}{2\pi}\int_{-\infty}^{\infty} |\alpha_\omega|^2 \, d\omega$$

Can easily do integral and use to give:

$$\boxed{\overline{|\eta_\omega|^2} = 2kT\gamma}$$

Fluctuation - Dissipation Theorem.

The Langevin Action

Action for random force $\eta(t)$ is

$$A[\eta_\omega] = \frac{1}{2}\sum_\omega \frac{|\eta_\omega|^2}{\overline{|\eta_\omega|^2}} = \frac{1}{4\gamma kT}\sum_\omega |\eta_\omega|^2$$

take I.F.T to get:

$$A[\eta(t)] = \frac{1}{4\gamma kT}\int \eta^2(t)\,dt$$

use ↑:

$$\therefore A[x(t)] = \frac{1}{4kT\gamma}\int (\kappa x + \gamma\dot{x} + m\ddot{x})^2 dt$$

take F.T. to get:

$$A[x_\omega] = \frac{1}{4\gamma kT}\sum \frac{|x_\omega|^2}{|\alpha_\omega|^2} = \sum_\omega \frac{a_\omega}{2}|x_\omega|^2$$

with $a_\omega = \frac{1}{2\gamma kT|\alpha_\omega|^2}$

now get $\underline{G-L \text{ action with } m=0}$

a Wiener process with restoring force

$\int x\dot{x}\,dt = d\left(\frac{x^2}{2}\right)$ is boundary term.

with $a = \frac{\kappa^2}{2kT\gamma}$ $\quad b=0$ $\quad c = \frac{\gamma}{2kT}$

With $\kappa = 0$, have G-L action for $v(t)$:

$$A[v(t)] = \frac{1}{4kT\gamma}\int \gamma^2 v^2 + m^2\dot{v}^2 \quad dt$$

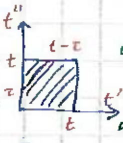
and the L. eq'n is now $\gamma v + m\dot{v} = \eta(t)$
or $(\gamma - i\omega m)v_\omega = \eta_\omega$

So from we get Lorentzian power spectrum,

$$\overline{|v_\omega|^2} = \frac{2kT\gamma}{\gamma^2 + m^2\omega^2} \Rightarrow \overline{v(\tau)v} = \overline{v^2}\, e^{-\frac{|\tau|}{\tau_v}}$$

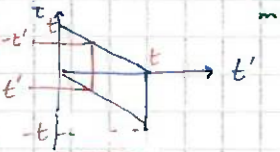
where $\tau_v = \frac{m}{\gamma}$ and $\overline{v^2} = \frac{kT}{m}$

$$\overline{x^2} = \int_0^t dt' \int_0^t dt'' \, \overline{v(t'')v(t')}$$

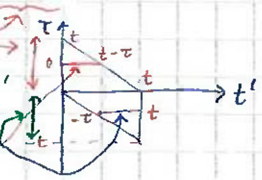


let $t'' = t' + \tau$

$$= \int_0^t dt' \int_{-t'}^{t-t'} d\tau \, \overline{v(\tau)v}$$



$$= \int_0^t d\tau \, \overline{v(\tau)v} \int_0^{t-\tau} dt' + \int_{-t}^0 d\tau \, \overline{v(\tau)v} \int_{-\tau}^t dt'$$



$$= 2\int_0^t d\tau \int_0^{t-\tau} dt' \, \overline{v(\tau)v} = 2\overline{v^2}\tau_v\left[t - \tau_v\left(1 - e^{-\frac{t}{\tau_v}}\right)\right]$$

$$\simeq \overline{v^2}t^2 \text{ for } t \ll \tau_v$$
$$\simeq 2\underbrace{\overline{v^2}\tau_v}_{D} t \text{ for } t \gg \tau_v$$

So in lim $\tau_v \to 0$ ie $m \to 0$, have Wiener process for $v(t)$.

Equation of State and Cumulants

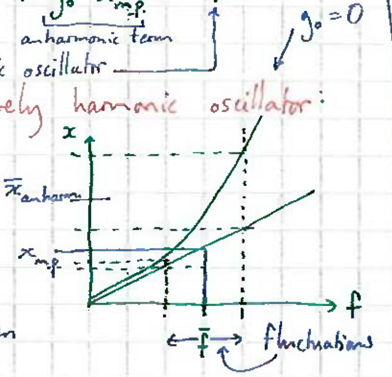
$$Z = \int dx \, \underbrace{e^{-\beta\left(\frac{k_0}{2}x^2 + \frac{g_0}{4}x^4 - xf\right)}}_{P(x)}$$

minimise exponent to give $x_{\text{most probable}}$ - ie what we would get if ∄ ~~fluct~~ fluctuations, randomness (in f ?)

$$\longrightarrow \quad k_0 x_{m.p.} + g_0 x_{m.p.}^3 = f$$

↑ harmonic oscillator; $g_0 x_{m.p.}^3$: anharmonic term

There is no difference between $\overline{x}$ and $x_{most.\,prob.}$ for purely harmonic oscillator:



Now, $$\boxed{\overline{x} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial f}}$$

let Z_0 be partition f'n for $f=0$ and $(\ )_0$ mean "average when $f=0$".

~~then~~, $$Z = Z_0 \overline{\left(e^{\beta x f}\right)}_0 = Z_0 \sum_{n=0}^{\infty} \frac{(\beta f)^n}{n!}\overline{x^n}$$ by def'n.

So:
$$= Z_0 \exp\left(\sum_{n=0}^{\infty}\frac{(\beta f)^n}{n!}\left(\overline{x_c^n}\right)_0\right)$$
$$\ln Z = \ln Z_0 + \sum_{n=0}^{\infty}\frac{(\beta f)^n}{n!}\left(\overline{x_c^n}\right)_0$$

$$\Rightarrow \boxed{\overline{x} = \sum_{n=0}^{\infty}\beta^{n-1}\frac{nf^{n-1}}{n!}\left(\overline{x_c^n}\right)_0}$$

Equation of State expansion of $\overline{x}$ in powers of f - coeffs are cumulants !

Perturbation Theory - Finding the Cumulants

We expand the cumulants as a power series in g_0

Let < > mean average w.r.t Gaussian problem ie $g_0 = f = 0$

Then:
$$Z_0 = Z_G \left\langle e^{-\frac{\beta g_0 x^4}{4}} \right\rangle = Z_G \exp\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{\beta g_0}{4}\right)^n}{n!}\langle u^n\rangle_c\right)$$

let $u = x^4$

now from the form of Z_0,

$$\boxed{\left(\overline{x_c^{2n}}\right)_0 = \left(-\frac{2}{\beta}\right)^n \frac{\partial \ln Z_0}{\partial k_0^n}}$$

and eg
$$\langle u^2\rangle_c = \langle u^2\rangle - \langle u\rangle^2$$
$$= \langle x^8\rangle - \langle x^4\rangle^2$$

Gaussian Problem so $\langle x^{2m}\rangle = \frac{(2m)!}{2^m m!}\sigma^{2m}$

can read off $\sigma^2 = \frac{1}{\beta k_0}$

So $\overline{x}$ is an ∞ series in f where each coeff is given in terms of an ∞ series of $\langle u^n\rangle_c$ which is given in terms of ie the cumulant expansion.

IPS ⑥

Self Consistent Renormalisation

Imagine that a random force $\eta(t)$ drives the fluctuations $f(t) + \eta(t) = k_0 x(t) + g_0 x^3(t)$

Now average with $P(x)$:

$$f = k_0 \bar{x} + g_0 \overline{x^3}$$

now $\overline{x^3} = \bar{x}^3 + 3\bar{x}\,\overline{x_c^2} + \overline{x_c^3}$

$$\therefore \qquad \% \qquad \lambda$$

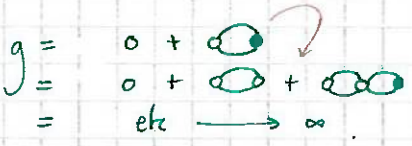
can find cumulants as f'ns of $\bar{x}$ which in turn is f'n of f. Let:

$$f = k\bar{x} + g\bar{x}^3 + \ldots\ldots$$

Then compare with to find self consistent expansions for k and g: $\left(\alpha = \frac{1}{k}\right)$

$$k = k_0 + 3\beta g_0 \alpha - 6\beta^2 g_0 \alpha^3 g$$

and iterate.



Green Function for the Schrödinger Equation.

G.F. is solution of S.E. which is zero for $t' > t$ and tends to $\delta(x - x')$ as $t' \to t$ from above.

So it propagates Ψ forward: $\Psi(x,t) = \int G(x,t;x',t')\,\Psi(x',t')\,dx'$ (for $t \geqslant t'$)

Assume $\Psi(x',t') = \delta(x' - y')$ then $\Psi(x,t) = G(x,t;y',t')$

ie G is amplitude of go from y' at $t' \to x$ at t.

Comparison with Wiener Process

Schröd. eq'n is: $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$

In the limit $\lambda \to 0$, the solution is $\Rightarrow$ $\frac{\partial \Psi}{\partial t} = \underbrace{\frac{\hbar i}{2m}}_{D} \frac{\partial^2 \Psi}{\partial x^2} - \underbrace{\left(\frac{-iV}{\hbar}\right)}_{\lambda}\Psi$

ie have a Diffusion eq'n with "local decay but coeffs are now imaginary

$D = \frac{i\hbar}{2m}$

$\lambda = -\frac{iV}{\hbar}$

$$\underbrace{G(\Delta x, \Delta t) = \frac{e^{-\Delta x^2/4D\Delta t}}{\sqrt{4\pi D \Delta t}}}_{\text{Solution to Diffusion equation.}} \quad \text{and F.T.} \Rightarrow G_q(\Delta t) = e^{-Dq^2\Delta t}$$

assume for v. small Δt, the two are independent, then,

$$G(\Delta x, \Delta t) = \frac{e^{-\left(\frac{\dot{x}^2}{4D} + \lambda\right)\Delta t}}{\sqrt{4\pi D \Delta t}}$$

$\dot{x} = \frac{\Delta x}{\Delta t}$ $\quad \Delta t = t - t'$

In limit $D \to 0$, G decays independently of pos'n

As before, can rewrite this as a path integral:

$$G_{ab} = \int_{ab} \underbrace{e^{\frac{i}{\hbar}\underbrace{\int_a^b \underbrace{\left(\frac{m\dot{x}^2}{2} - V(x)\right)}_{L} dt}_{S[x(t)]}} \frac{dx_2 \cdots dx_{n-1}}{(4\pi D \Delta t)^{\frac{n-1}{2}}}}$$

↳ this is prob. amplitude of going from $a \to b$ via path $x(t)$.

in lim $\Delta t \to 0$, get: $\to \int e^{\frac{i}{\hbar}S[x(t)]}\,\mathcal{D}x(t)$

G_{ab} is a conditional relative amplitude ie prob $\propto |\Psi_b|^2$ where $\Psi_b = \int G_{ab}\Psi_a\,dx_a$

In limit $D \to 0$ ie $\frac{i\hbar}{2m} \to 0$, fluctuations $\to 0$.

$\quad \longrightarrow$ E-L eq'ns.

For $D \neq 0$, must sum over all paths $\to$ interference $\to$ Q.M.

$\exists$ maths probs in the complex case + spin (eg) not included.

Perturbation Theory for Q.M.

Harmonic Oscillator is exactly solvable.

Otherwise - need P.T. - expand a perturbing potential $V(x(t), t)$:

let $U(t) = \frac{V}{i\hbar}$ (λ from)

then $\frac{i}{\hbar}S \to \frac{i}{\hbar}S + \int_a^b U\,dt$

(unperterbed motion - Gaussian - $\frac{m\dot{x}^2}{2}$)

Now $G_{ab} = \int_{ab} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S}\, e^{\int U dt}$

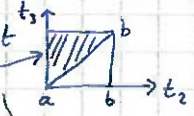
$e^{\int U dt} \to 1 + \int U(t_2)\,dt_2 + \frac{\int U(t_2)\,U(t_3)\,dt_2\,dt_3}{2!} + \ldots$ $\quad 2! \longleftarrow$ coz exp.

"free propagation" from $a \to b$ ie no potential.

$$g_{ab} = \int g_{a2}\,g_{2b}\,dx_b$$

$= $ etc etc

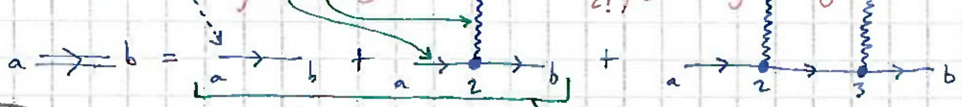
can let $t_3 > t_2$ or relax constraint as want



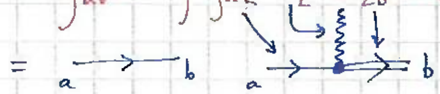
BORN APPROXIMATION is stopping the series after one interaction.

$t_3 > t_2$

$$G_{ab} = g_{ab} + \int g_{a2}\,U_2\,g_{2b}\,dx_2\,dt_2 + \frac{2!}{2!}\int g_{a2}\,U_2\,g_{23}\,U_3\,g_{3b}\,dx_2\,dt_2\,dx_3\,dt_3 + \ldots$$

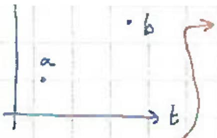


$$= g_{ab} + \int g_{a2}\,U_2\,G_{2b}\,dx_2\,dt_2$$



TP3 ⑦

Moments and Q.M.



In Q.M. $1 = \int \psi_b \psi_b^* dx_b$

but $\psi_b = \int G_{ab} \psi_a dx_a$

$$\Rightarrow \int G_{ab} \psi_a \psi_b^* dx_a dx_b = 1$$

$$\Rightarrow \int D x(t) \underbrace{\psi_a e^{\frac{i}{\hbar} S[x(t)]} \psi_b^*}_{} = 1$$

all paths from a → b and all a, b.

this looks like prob. of taking path $x(t)$ from $a \rightarrow b$.

Prob depends on $\psi(x_a, t_a)$ and $\psi^*(x_b, t_b)$ and the amp. $\exp\left(\frac{i}{\hbar} S\right)$.

FIRST MOMENT (at t_2)

$$\overline{x(t_2)} = \int^{t_2} x(t_2) P[x(t)] D x(t)$$

$$= \int \underbrace{\psi_a G_{a2}}_{} x_2 \underbrace{G_{2b} \psi_b^*}_{(G_{b2} \psi_b)^* \text{ (advanced)}} dx_a dx_2 dx_b$$

$$= \int \psi_2 x_2 \psi_2^* dx_2$$

$$= \langle x \rangle_{t_2} \text{ or } \langle x(t_2) \rangle$$

↑ S.P. ↑ H.P.

ie averaging over $P[x(t)]$ is same as usual average.

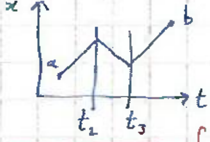
the general case:

n-point correlation function:

$$\overline{x(t_n) \dots x(t_1)} = \langle T(\hat{x}(t_n) \cdots \hat{x}(t_1)) \rangle$$

SECOND MOMENT (at t_2, t_3 $t_3 > t_2$)

called 2 pt correlation function.



$$\overline{x(t_3) x(t_2)} = \int x(t_2) x(t_3) P[x(t)] D x(t)$$

$$= \int \underbrace{\psi_a G_{a2}}_{} x_2 G_{23} x_3 \underbrace{G_{3b} \psi_b^*}_{} dx_a dx_2 dx_3 dx_b$$

$$= \int \psi_2 x_2 G_{23} x_3 \psi_3^* dx_2 dx_3$$

cf. usual Q.M. $\langle x_3 x_2 \rangle = \underbrace{\sum_n \langle \psi_3 | \hat{x} | n_3 \rangle \langle n_2}_{G_{23}} | \hat{x} | \psi_2$

but time ordering is important because ops $\hat{x}$ do not nec. commute:

$$\overline{x(t_3) x(t_2)} = \langle x_3 x_2 \rangle \text{ if } t_3 > t_2$$
$$= \langle x_2 x_3 \rangle \text{ if } t_2 > t_3.$$

Propagator and energy eigenstates

We know $\psi(x,t) = \int G(x, x', t) \psi(x') dx'$

let $t' = 0$

now expand $\psi(x,t) = \sum_n c_n \phi_n(x) e^{-i\omega_n t}$

↑ energy e-states

where $c_n = \langle n | \psi \rangle$

ie $c_n = \int \phi_n^*(x') \psi(x') dx'$

So

$$\sum_n \int \phi_n^*(x') \psi(x') dx' \phi_n(x) e^{-i\omega_n t} = \int G \psi(x') dx'$$

$$\text{ie } \boxed{G(x, x', t) = \sum_n \phi_n(x) \phi_n^*(x') e^{-i\omega_n t}}$$

Involves all e-states

for $t > 0$

Can get density of states

$$\underline{\underline{\operatorname{Tr} G}} = \int_{-\infty}^{\infty} dx\, G(x, x, t) = \underline{\underline{\sum_n e^{-i\omega_n t}}}$$

now time F.T. of Tr G

= Tr of FT(G)

now F.T. $e^{-i\omega_n t}$ for $t > 0$ is $i \int_0^\infty e^{i(\omega - \omega_n + i\epsilon)t} dt = \frac{-1}{(\omega - \omega_n + i\epsilon)}$

↑ decay

$$= \frac{-1}{\omega - \omega_n} + i\pi \delta(\omega - \omega_n)$$

$$\text{So } \boxed{\frac{1}{\pi} \operatorname{Im}\left[\operatorname{Tr}(i G_\omega)\right] = \sum_n \delta(\omega - \omega_n)}$$

Relation to Partition Function

We have $\operatorname{Tr} G = \sum_n e^{-i\omega_n t}$ now let $t = \frac{\tau_0}{i} = \frac{-i\hbar}{kT}$

Then have $\operatorname{Tr} G = \sum_{\text{all } n} e^{-\frac{\hbar \omega_n}{kT}} = Z$

$$\text{ie } \boxed{Z = \operatorname{Tr} G\left(x, x, \frac{\tau_0}{i}\right)}$$

So can express Z as a path integral from position x at $t = 0$ to position x at $t = \tau_0 / i$, summed over all x

the action is now

$$S = \int_0^{\tau_0/i} dt \left(\frac{m}{2} \dot{x}^2 - V(x)\right)$$

let $t = \frac{\tau}{i}$, $y(\tau) = x(t)$, $\dot{y} = \frac{dy}{d\tau}$

$$\text{then } \frac{i}{\hbar} S = -\frac{1}{\hbar} \underbrace{\int_0^{\tau_0} \left(\frac{m \dot{y}^2}{2} + V(y)\right) d\tau}_{\text{Euc. action}}$$

$$\text{then } Z = \int_{\substack{\text{all paths} \\ \text{from } x \to x}} e^{-\frac{S_{Euc}[x(\tau)]}{\hbar}} D x(\tau)$$

summing also over all possible values of the starting / ending pts x.

Examples

① For a free particle, $G(x, x', t) = \frac{e^{-\frac{\Delta x^2}{4Dt}}}{\sqrt{4\pi D t}}$ where $D = \frac{i\hbar}{2m}$

so with $t = \frac{\tau_0}{i} = \frac{\hbar}{ikT}$ and $x = x'$ so $\Delta x = 0$,

we get:

$$Z_{\text{free particle}} = \sqrt{\frac{mkT}{2\pi\hbar^2}} \left(= \int \frac{dk}{2\pi} e^{-\beta \frac{p^2}{2m}} \text{ in norm. Stat. mech.} \right)$$

↑ no. of states per unit k...

↑ gaussian

② Harmonic oscillator, $V = \frac{1}{2} m \Omega^2 x^2$

$$Z_{\text{sho}} = \frac{1}{2 \sinh\left(\frac{\hbar \Omega}{2kT}\right)}$$

for high temp, ↔ small τ_0, let $V(x(0)) = V($

$$\text{then } Z = \sqrt{\frac{mkT}{2\pi\hbar^2}} \underbrace{\int_{-\infty}^{\infty} dx \exp\left(-\frac{V(x)}{kT}\right)}_{\text{classical result}}$$

QM deg. factor.

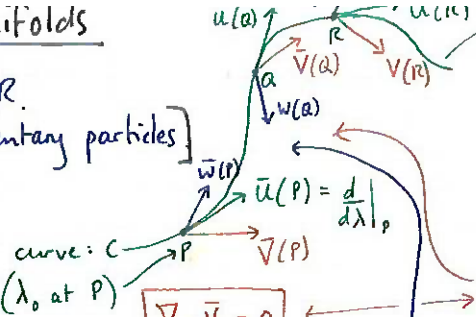
then Q.M. correction $O\left(\frac{1}{T}\right)$ is

$$V(x) \rightarrow V(x) + \frac{\hbar^2}{24 mkT} \frac{\partial^2 V(x)}{\partial x^2} \dots$$

Connections for Riemannian Manifolds

Connections : Affine connections ⟶ G.R.
[Gauge connections ⟶ elementary particles]

Covariant Derivative



~~curve C created by Parallel transporting $\bar{u}$ from point P - using the affine conn.~~

no - that ⤴ is all wrong.

$\boxed{\bar{u} \text{ is just the tangent vector to } C}$

Given $\bar{V}(P)$, can define vector field $\bar{V}$ along curve by parallel transport.

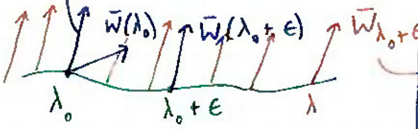
$$\boxed{\nabla_{\bar{u}} \bar{V} = 0}$$

PARALLEL TRANSPORT.

Given another vector field $\bar{W}$ in M, it will not nec. be parallel transport-ed along C.

$\bar{W}_{\lambda_0+\epsilon}(\lambda_0)$

$$\boxed{\left(\nabla_{\bar{u}} \bar{W}\right)_P = \lim_{\epsilon \to 0} \frac{\bar{W}_{\lambda_0+\epsilon}(\lambda_0) - \bar{W}(\lambda_0)}{\epsilon}}$$



$\boxed{\text{have generated a vector by } \nabla_{\bar{u}} \bar{W}(\lambda_0+\epsilon) = 0}$

evaluate this vector at λ_0
then subtract
$\bar{w}$ already there!

For scalar function:
Parallel transport gives: $f_{\lambda_0+\varepsilon}(\lambda) = f(\lambda_0+\epsilon)$

$$\Rightarrow \boxed{\left(\nabla_{\bar{u}} f\right)_P} = \lim_{\varepsilon \to 0} \frac{f(\lambda_0+\varepsilon) - f(\lambda_0)}{\varepsilon} = \boxed{\frac{df}{d\lambda}\Big|_{\lambda_0}}$$

Product rules :

$$\nabla_{\bar{u}}(\bar{A} \otimes \bar{B}) = \nabla_{\bar{u}} \bar{A} \otimes \bar{B} + \bar{A} \otimes \nabla_{\bar{u}} \bar{B}$$

$$\nabla_{\bar{u}} \langle \tilde{\omega}, \bar{A} \rangle = \langle \nabla_{\bar{u}} \tilde{\omega}, \bar{A} \rangle + \langle \tilde{\omega}, \nabla_{\bar{u}} \bar{A} \rangle$$

Changing Parameter

Suppose we let $\lambda \to \mu$ then $\bar{u}' = \frac{d\lambda}{d\mu} \bar{u}$ (by chain rule)

⟶ new tangent vector $\frac{d}{d\mu}$

Parallel transport is by def'n indep. of choice of parameter

But $\varepsilon \to \delta\mu = \frac{d\mu}{d\lambda} \varepsilon$ as $(\varepsilon \equiv \delta\lambda)$ $\Rightarrow \boxed{\nabla_{g\bar{u}} \bar{W} = g \nabla_{\bar{u}} \bar{W}}$

also, $(\nabla_{\bar{u}} \bar{W})_P + (\nabla_{\bar{V}} \bar{W})_P = \nabla_{\bar{u}+\bar{v}} \bar{W}$

So together, : $\nabla_{f\bar{u}+g\bar{v}} \bar{W} = f \nabla_{\bar{u}} W + g \nabla_{\bar{V}} \bar{W}$

So $\nabla_{\bar{u}}$ behaves like $\underline{\nabla}$ of vector calculus.

Components

Any tensor can be written as a sum of basis tensors
can be derived from vector basis $\{\bar{e}_i\}$

So, connection can be completely described by giving:

$$\boxed{\nabla_{\bar{e}_i} \bar{e}_j = \Gamma^k{}_{ji} \bar{e}_k}$$

CHRISTOFFEL SYMBOLS

$(\nabla_{\bar{e}_i} \equiv \nabla_i)$

So
In an n dimensional manifold, the n^3 f'ns $\Gamma^k{}_{ji}$ completely determine the affine connection.

So: $\nabla_{\bar{u}} \bar{V} = u^i \nabla_{\bar{e}_i} (V^j \bar{e}_j)$

~~chain~~ product rule

$= u^i (\nabla_i V^j) \bar{e}_j + u^i V^j \nabla_{\bar{e}_i} \bar{e}_j$

$(\nabla_{u^i \bar{e}_i} V^j)$ (f'n)

$= \frac{dV^j}{d\lambda} \bar{e}_j + u^i V^j \Gamma^k{}_{ji} \bar{e}_k$

relable indices.

$= \left(\frac{dV^j}{d\lambda} + \Gamma^j{}_{ki} V^k u^i\right) \bar{e}_j$

Since $\nabla \bar{V}$ is a tensor it has components: (remove u^i from)

$$\boxed{(\nabla \bar{V})^j{}_i = V^j{}_{,i} + \Gamma^j{}_{ki} V^k = V^j{}_{;i}}$$

For an $\binom{N}{M}$ tensor, T,

$$T^{i \ldots j}{}_{k \ldots l;m} = T^{i \ldots j}{}_{k \ldots l,m} + \Gamma^i{}_{nm} T^{n \ldots j}{}_{k \ldots l} + \Gamma^j{}_{nm} T^{i \ldots n}{}_{k \ldots l} - \Gamma^n{}_{km} T^{i \ldots j}{}_{n \ldots l} - \Gamma^n{}_{lm} T^{i \ldots j}{}_{k \ldots n}.$$

Geodesics

- is a curve that parallel transports its own tangent vector:

$$\boxed{\nabla_{\bar{u}} \bar{u} = 0}$$

ie $\frac{du^i}{d\lambda} + \Gamma^i{}_{jk} u^j u^k = 0$

then with $u^i = \frac{dx^i}{d\lambda}$

$$\Rightarrow \boxed{\frac{d^2 x^i}{d\lambda^2} + \Gamma^i{}_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0}$$

⤴ a quasi-linear set of differential equations for $x^i(\lambda)$ - the curve.

- if change parameter to one that leaves the form of eq'n invariant then the old + new parameters are both affine params. $\lambda \to \mu = a\lambda + b$ does the trick, a, b const.

Riemann Tensor

$$[\nabla_{\bar{u}}, \nabla_{\bar{v}}] - \nabla_{[\bar{u}, \bar{v}]} \equiv R(\bar{u}, \bar{v})$$

is a multiplicative operator taking a vector → vector, ∴ $R(\bar{u}, \bar{v})$ is a $\binom{1}{1}$ tensor. Regarding $\bar{u}$ and $\bar{v}$ as variable arguments, R is then a $\binom{1}{3}$ tensor, with components:

$$R^l{}_{kij} \bar{e}_l = [\nabla_i, \nabla_j] \bar{e}_k - \nabla_{[\bar{e}_i, \bar{e}_j]} \bar{e}_k$$

$\Rightarrow$ In a coord. basis $([\bar{e}_i, \bar{e}_j] = 0)$ $R^l{}_{kij} = \Gamma^l{}_{kj,i} - \Gamma^l{}_{ki,j} + \Gamma^m{}_{kj} \Gamma^l{}_{mi} - \Gamma^m{}_{ki} \Gamma^l{}_{mj}$

In a non coordinate basis, let $[\bar{e}_j, \bar{e}_k] = C^i{}_{jk} \bar{e}_i$
-get extra term, $-C^m{}_{ij} \Gamma^l{}_{km}$

If symetrise last 2 indices : $R^l{}_{k(ij)} = \frac{1}{2}(ij + ji)$

$\Rightarrow$ get zero : $R^l{}_{k(ij)} = 0$ - antisym in i, j.

Also, $R^l{}_{[kij]} = 0$ and : $\boxed{R^l{}_{k[ij;m]} = 0}$

BIANCHI IDENT.

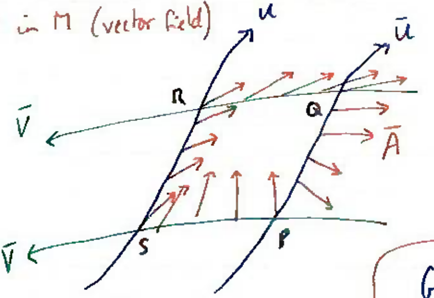
in a coord. basis, these ⤴ are equivalent to the JACOBI IDENTITY

Connections for Riemannian Manifolds (cont'd)

Geometric Interpretation of Riemann Tensor

$\bar{u}, \bar{v}$ are congruences, $[\bar{u},\bar{v}]=0$
So RQPS is a closed loop.

ie $\bar{A}$ exist in M (vector field)



- go round loop PQRSP as shown.
Change in $\bar{A}$ is

$$\boxed{\delta A^i = \underbrace{\lambda\mu}_{\text{"area"}} R^i{}_{jkl} A^j U^k V^l}$$

→ need to parallel transport A say from $Q \to P$:

$$\bar{A}(\text{at } P \text{ from } Q) = \bar{A}(P) + \lambda \nabla_{\bar{u}} \bar{A}(P) + \tfrac{1}{2}\lambda^2 \nabla_{\bar{u}}\nabla_{\bar{u}}\bar{A}(P)$$

$$= e^{\lambda\nabla_{\bar{u}}} \bar{A}\Big|_P$$

- take $\bar{A}$ from $(R \to Q \to P)$

get: $\bar{A}_{R\to Q\to P} = \left(e^{\lambda\nabla_{\bar{u}}} e^{\mu\nabla_{\bar{v}}} \bar{A}\right)_P$

- take $\bar{A}$ from $(R \to S \to P)$

get: $\bar{A}_{R\to S\to P} = \left(e^{\mu\nabla_{\bar{v}}} e^{\lambda\nabla_{\bar{u}}} \bar{A}\right)_P$

call difference, : $\delta\bar{A} = [e^{\lambda\nabla_{\bar{u}}}, e^{\lambda\nabla_{\bar{v}}}]\bar{A}$

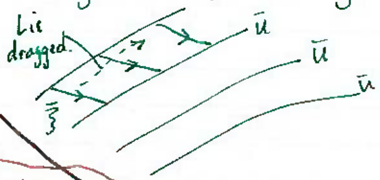
$$= \lambda\mu[\nabla_{\bar{u}}, \nabla_{\bar{v}}]\bar{A} + O(\text{cubed...}) \to 0$$

Because $[\nabla_{\bar{u}}, \nabla_{\bar{v}}]$ is Riemann tensor when $[\bar{u},\bar{v}]=0$ ...... get

So Riemann Tensor measures change in vector when transported round closed loop.

GEODESIC DEVIATION:

- the measure of how geodesics started parallel do not stay parallel:
Congruence $\bar{u}$, connecting vector, $\bar{\xi}$:



→ Let $\bar{u}$ be a geodesic congruence.
know: $\underline{\nabla_{\bar{u}}\bar{u}=0}$ and $\underline{\pounds_{\bar{u}}\bar{\xi}=0}$

Now $\nabla_{\bar{u}}\bar{\xi}$ contains info about if init parallel or not.
Second deriv. tells us about how $\bar{\xi}$ changes as it is lie dragged: $\nabla_{\bar{u}}\nabla_{\bar{u}}\bar{\xi}$

$$(\pounds_{\bar{u}}\tilde{\omega})_i = \omega_{i;j}u^j + \omega_j u^j{}_{,i}$$

can change all commas to semicolons.

$$= \omega_{i;j}u^j + \omega_j u^j{}_{;i}$$

$$= \nabla_{\bar{u}}(\cancel{\pounds_{\bar{u}}\bar{\xi}} + \nabla_{\bar{\xi}}\bar{u}) \quad 0$$

$$= \nabla_{\bar{u}}\nabla_{\bar{\xi}}\bar{u}$$

$$= \underbrace{[\nabla_{\bar{u}}, \nabla_{\bar{\xi}}]\bar{u}}_{R(\bar{u},\bar{\xi})\bar{u}} + \nabla_{\bar{\xi}}\overset{0}{\nabla_{\bar{u}}\bar{u}}$$

because: $u^j{}_{;k}u^k=0$

$$\Rightarrow (\xi^i{}_{;j}u^j)_{;k}u^k = R^i{}_{jkl}u^j u^k \xi^l$$

$$\Rightarrow \boxed{\xi^i{}_{;j;k}u^j u^k = R^i{}_{jkl}u^j u^k \xi^l}$$

EQUATION OF GEODESIC DEVIATION

Flat Spaces

From eq'n of geod. deviation, can see that parallel lines remain so when extended iff $\underline{R=0}$

So $R(\sim;-,-,-)$ measures curvature of a space with a connection - don't need metric! (In a flat space M, T_p is M!)
On a flat space, coords exist where Christoffel symbols vanish
however - ∃ coords where $\Gamma^i{}_{jk}$ don't all vanish - eg sph. polars.

Metric Connections

- If have affine connection + metric on M, then in order to garantee compatability, the inner prod. of two vectors must be invariant under parallel transport: $\Rightarrow \nabla g_1 = 0$

This in turn implies that:

$$\boxed{\Gamma^i{}_{jk} = \tfrac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})}$$

So a metric determines connection uniquely.

$\Rightarrow$ if in normal coords (ie $\Gamma^i{}_{jk}$) at P, then at P

$$\boxed{g_{lm,n}=0}$$

So, using components of R in coord basis,

$$R_{ijkl} = g_{im}R^m{}_{jkl} = \tfrac{1}{2}(g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik})$$

in a normal coord. system.

→ So coz sym of metric,

$$\boxed{R_{ijkl} = R_{klij}}$$

Also, $(\pounds_{\bar{v}}g_1)_{ij} = \nabla_i V_j + \nabla_j V_i$

So a Killing Vector obey's Killing's Eq'n:

$$\nabla_i V_j + \nabla_j V_i = 0$$

Ricci tensor is obtained through contraction of $R^i{}_{kjl}$:

$$\boxed{R_{kl} = R^i{}_{kil}}$$

symmetric

Ricci Scalar is: $\boxed{R = g^{kl}R_{kl}}$

Then define Einstein Tensor by:

$$\boxed{G^{ij} \equiv R^{ij} - \tfrac{1}{2}Rg^{ij}}$$

If contract the Bianchi Identities,
ie $R^i{}_{j[il;m]}=0$ and $g^{jl}R^i{}_{j[il;m]}=0$

Then get: $\boxed{G^{ij}{}_{;j}=0}$

→ Einstein (source free) eq'ns are: $\boxed{G^{ij}=0}$

→ This constraint reduces number of independent equations from 10 to 6 thus meaning that the solution, g_{ij}, which also has 10 indep. components, is only determined up to the 4 functional degrees of freedom represented by the coordinate transformations of g_{ij}.

Define Weyl Tensor: (for some reason...)

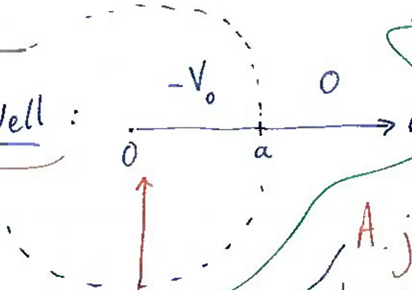
$$C^{ij}{}_{kl} = R^{ij}{}_{kl} - 2\delta^{[i}{}_{[k}R^{j]}{}_{l]} + \tfrac{1}{3}\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}R$$

Principle of Equivalence

It is natural to a assume that the laws of Physics have same equations in curved space as flat - they involve $\Gamma^i{}_{jk}$ but not the Riemann Tensor. This the principle of minimal coupling (of fields to curvature) or strong equivalence principle. It is just a result of flat space being no more or less basic / fundamental than curved space - they both have connection and a metric.

Scattering Theory

Spherically symmetric Square Well :



So for $r \leqslant a$ particle is described by $\psi(\underline{r})$ where:

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\underline{r}) = E\,\psi(\underline{r})$$

Now : if $V=0$

↓

free particle case.

Only Bessel f'ns are regular at origin and assymptotic limit is

$$\boxed{j_l(kr) = \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr}}$$

COMPARE

to find that $\delta_l(k)$ is a PHASE SHIFT

if V is as above get $R_l(r) = A' j_l(qr)$ inside
$= B j_l(kr) + C n_l(kr)$ outside

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

$q = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}$

Spherical symmetry

$\Rightarrow$ use sep. of variables :

$$\psi(\underline{r}) = R(r)\, Y_{lm}(\theta, \phi)$$

$A.\, j_l(kr) + B\, n_l(kr)$ Spherical Harmonics

spherical Bessel f'n spherical Neuman f'n

because $R(r)$ obeys $\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{l(l+1)}{\rho^2} + 1\right] R_l(\rho) = 0$ whose sol'ns are j_l and n_l when $k = \sqrt{\frac{2m}{\hbar^2}(E-V)}$ and $\rho = kr$

Get A, B, C from B.C.'s (continuity) and so assymp. form is :

$$R_l(r) \simeq \frac{B}{kr}\left[\sin\left(kr - \frac{l\pi}{2}\right) - \frac{C}{B}\cos\left(kr - \frac{l\pi}{2}\right)\right]$$

Then can put $R_l(r)$ into different form by putting $\frac{C}{B} = -\tan \delta_l(k)$

to get

$$\boxed{R_l(r) \xrightarrow[r\to\infty]{} \frac{B}{\cos\delta_l(k)} \cdot \frac{\sin\left(kr - \frac{l\pi}{2} + \delta_l(k)\right)}{kr}}$$

Spherical Neuman and Bessel Functions

Subst $k = \left(\frac{2m}{\hbar^2}(E-V)\right)^{1/2}$ and $\rho = kr$ and $\psi = R_l Y_{lm}$

to get : $\left(\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{l(l+1)}{\rho^2} + 1\right) R_l(\rho) = 0$ } Solutions are :

$$\boxed{\begin{aligned} j_l(\rho) &= (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\sin\rho}{\rho}\right) \\ n_l(\rho) &= -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\cos\rho}{\rho}\right) \end{aligned}}$$

Can both be expressed as power series in ρ.

The SPHERICAL HANKEL f'ns are :

$$\left.\begin{aligned} h_l^{(1)}(\rho) &= j_l(\rho) + i n_l(\rho) \\ h_l^{(2)}(\rho) &= j_l(\rho) - i n_l(\rho) \end{aligned}\right] \text{So } h_l^{(2)}(\rho) = \left(h_l^{(1)}(\rho)\right)^*$$

EXPLICIT FORMS :

$j_0 = \frac{\sin\rho}{\rho}$	$n_0 = -\frac{\cos\rho}{\rho}$	$h_0^{(1)} = \frac{e^{i\rho}}{i\rho}$
$j_1 = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$	$n_1 = \frac{-\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$	$h_1^{(1)} = -\frac{e^{i\rho}}{\rho}\left(1 + \frac{i}{\rho}\right)$

NB : j_l is regular at origin , n_l is singular at origin.

$\exists$ recursion formulae for j, n, h.

and j_l has an integral representation:

$$j_l(\rho) = \frac{(-i)^l}{2}\int_{-1}^{1} dz\, P_l(z)\, e^{i\rho z}$$

Assymptotic Behaviour

Small arguments ($\rho \ll 1$)

$$j_l(\rho) \rightarrow \frac{\rho^l}{(2l+1)!!}$$

$$n_l(\rho) \rightarrow -\frac{(2l-1)!!}{\rho^{l+1}}$$

Large arguments ($\rho \gg 1$)

$$j_l(\rho) \rightarrow \frac{1}{\rho}\sin\left(\rho - \frac{l\pi}{2}\right)$$

$$n_l(\rho) \rightarrow -\frac{1}{\rho}\cos\left(\rho - \frac{l\pi}{2}\right)$$

$$h_l^{(1)}(\rho) \rightarrow \frac{-i}{\rho}\, e^{i\left(\rho - \frac{l\pi}{2}\right)}$$

Expanding plane waves in sph. harmonics and sph. Bessel f'ns

The ψ_{lm} are a complete set

$\therefore$ can expand $e^{i\underline{k}\cdot\underline{r}}$ in terms of ψ_{lm} $(= R_l Y_{lm})$ with coeffs $c_{lm}(\underline{k})$:

$$\boxed{e^{i\underline{k}\cdot\underline{r}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}(\underline{k})\, j_l(kr)\, Y_{lm}(\theta, \phi)}$$

Use azimuthal Symmetry : $\underline{k}\cdot\underline{r} = kr\cos\theta$ in sph. polars

So $c_{lm} \rightarrow A_l$ (no m) $Y_{lm} \rightarrow P_l(\cos\theta)$ with suitable norm. : $\left(\frac{2l+1}{4\pi}\right)^{1/2}$

Use orthonormality of Legendre Pol. to get A_l :

$$A_l\, j_l(kr) = \frac{1}{2}\sqrt{4\pi(2l+1)}\int_{-1}^{1} dz\, P_l(z)\, e^{ikrz}$$

Compare to realise that

$A_l = i^l \sqrt{4\pi(2l+1)}$ So we have it ! $\boxed{e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)}$

Now , can rewrite so as to generalise for arbitrary $\underline{k}$ using :

$$\boxed{P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\Omega_{\underline{k}})\, Y_{lm}(\Omega_{\underline{r}})}$$

ADDITION THEOREM for SPHERICAL HARMONICS

$\Omega_{\underline{k}} \equiv (\theta_{\underline{k}}, \phi_{\underline{k}})$ etc

So :

$$\boxed{e^{i\underline{k}\cdot\underline{r}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l\, j_l(kr)\, Y^*_{lm}(\Omega_{\underline{k}})\, Y_{lm}(\Omega_{\underline{r}})}$$

Scattering Theory Cont'd ...

Setting up the problem

If scattering problem has incoming p. wave + outgoing sph. wave

$$\psi(\underline{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

Then f ≡ scattering amplitude

and $$\frac{d\sigma}{d\Omega} = |f|^2$$

Differential Cross Section:

$$\frac{d\sigma}{d\Omega}\cdot d\Omega = \frac{\text{no. of particles scattered into } d\Omega \text{ per sec.}}{\text{incident flux.}}$$

$$= r^2 d\Omega \frac{|j_{\text{scatt}}|^2}{|j_{\text{inc}}|^2}$$

total cross section:

$$\sigma_{\text{tot}} = \int_{\text{unit sphere}} \frac{d\sigma}{d\Omega}\, d\Omega$$

P.W.A. O.K. for when incident particle energy has low energy compared with scattering pot. in fact any energy will do - just need many partial waves

assume pot. is short ranged $\int_0^\infty r^2 V(r)dr < \infty$

but NOT NECESSARILY WEAK !

Partial Wave Analysis

Schrödinger Eq'n can be written as: $\boxed{\left(\nabla^2 + k^2 - U(r)\right)\psi = 0}$

where $U = \frac{2mV}{\hbar^2}$ $k^2 = \frac{2mE}{\hbar^2}$

Then can sep. var. by decomposing into sph. waves:

let $$\psi(\underline{r}) = \sum_{l=0}^{\infty} i^l(2l+1)\underbrace{R_l(r)}\, P_l(\cos\theta)$$

where R_l satisfies radial eq'n:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2 - U\right)R_l(r) = 0$$

but for small r R must $< \infty$ ∴ take lim $r \to 0$ get $R = Ar^l$

Anyway, incoming beam is e^{ikz}

$$= \underbrace{\sum_{l=0}^{\infty} i^l(2l+1)\, j_l(kr)\, P_l(\cos\theta)}_{\text{spherical waves}} = \sum_{l=0}^{\infty} i^l(2l+1)\,\frac{1}{2}\left(\overset{\text{INGOING}}{h_l^{(2)}(kr)} + h_l^{(1)}(kr)\right) P_l(\cos\theta)$$

OUTGOING

NOW: Scattering Potential can only affect outgoing waves, so assymptotic solution must be of form

$$\psi(r) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} i^l(2l+1)\frac{1}{2}\left(h_l^{(2)} + \underbrace{S_l(k)}_{\text{effect of potential.}} h_l^{(1)}\right) P_l(\cos\theta)$$

Can rewrite as:

$$e^{i\underline{k}\cdot\underline{r}} + \sum_{l=0}^{\infty} i^l(2l+1)\frac{1}{2}\left(S_l(k) - 1\right) h_l^{(1)}(kr) P_l(\cos\theta)$$

replace with assymptotic form to get

$$\psi(\underline{r}) \xrightarrow[r\to\infty]{} e^{i\underline{k}\cdot\underline{r}} + f(\theta)\frac{e^{i\underline{k}\cdot\underline{r}}}{r}$$

where

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left[S_l(k) - 1\right]P_l(\cos\theta)$$

(SCATTERING AMPLITUDE)

NOW: let $S_l(k) = e^{2i\delta_l(k)}$ ie write complex S in terms of real phase shift.

then $$\boxed{f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)\, P_l(\cos\theta)}$$

Resonance

total x-section, $\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega}d\Omega$

$$= \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

can be written as sum of partial cross sections

$$= \sum_l \sigma_l \quad \text{where } \sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

So, $\boxed{\sigma_l \leq \frac{4\pi}{k^2}(2l+1)}$, equal when $\boxed{\delta_l = \left(n+\frac{1}{2}\right)\pi}$

(elastic scattering ⇒ δ_l real)

UNITARY BOUND

CONDITION FOR RESONANCE

Optical Theorem

$\left(\text{NB } S_l = e^{2i\delta_l}\right)$

Consider Imaginary Part of $f(\theta)$:

$$\text{Im}\left[f(\theta)\right] = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\left[2\sin^2\delta_l\right]P_l(\cos\theta)$$

and $P_l(1) = 1$ ⇒ $\text{Im}\left[f(0)\right] = \left(\frac{4\pi}{k}\right)^{-1}$ total X-section

mp 5

ie $$\boxed{\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}\left[f(0)\right]}$$

OPTICAL THEOREM

Integral Form of S.E.

NB Q contains ψ !

Can write S.E. as the Helmholtz Eq. if $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $Q(\underline{r}) = \frac{2m}{k^2}V(\underline{r})\psi(\underline{r})$

(not necessarily spherically symmetric.)

Then : $$\boxed{\left(\nabla^2 + k^2\right)\psi(\underline{r}) = Q(\underline{r})}$$

INHOMOGENOUS SOURCE TERM

Now, find Green Function solution of H.E.

ie if $$\left(\nabla^2 + k^2\right)G(\underline{r}-\underline{r}') = \delta(\underline{r}-\underline{r}')$$

then $$\psi(\underline{r}) = \int_{\text{all space}} G(\underline{r}-\underline{r}')\,Q(\underline{r}')\,d^3\underline{r}'$$

Integral eq'n because

bear in mind $$\delta(\underline{r}-\underline{r}') = \frac{1}{(2\pi)^3}\int_{\text{all space}} e^{i\underline{s}\cdot(\underline{r}-\underline{r}')}\,d^3\underline{s}$$

but what is G? write as a Fourier Transform

ie $$G(\underline{r}-\underline{r}') = \frac{1}{(2\pi)^{3/2}}\int_{\text{all space}} e^{i\underline{s}\cdot(\underline{r}-\underline{r}')}\,g(\underline{s})\,d^3\underline{s}$$

and then subst into the Helmholtz Equation:

$$\left(\nabla^2 + k^2\right)G(\underline{r}) = \frac{1}{(2\pi)^{3/2}}\int_{\text{all space}}\left(k^2 - s^2\right)e^{i\underline{s}\cdot\underline{r}}\,g(\underline{s})\,d^3\underline{s}$$

(applying ∇^2 inside $\int$)

$$= \delta(\underline{r})$$

Now, integrands must be equal: $$g(\underline{s}) = \frac{1}{(2\pi)^{3/2}}\frac{1}{(k^2 - s^2)}$$

So $$G = -\frac{e^{i\underline{k}\cdot(\underline{r}-\underline{r}')}}{4\pi r}$$

(must use contour integration)

$$\Rightarrow \boxed{\psi(\underline{r}) = \psi_0(\underline{r}) - \frac{m}{2\pi\hbar^2}\int_{\text{all space}}\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}V(\underline{r}')\psi(\underline{r}')\,d^3\underline{r}'}$$

INTEGRAL FORM OF S.E.

where $\psi_0(\underline{r})$ satisfies homogenous eq'n ie it is the complimentary function.

Scattering Theory Cont'd...

Born Approximation

Scattering Potential must be:

WEAK — So that it can be treated as a perturbation

LOCALISED about $\underline{r}'=0$ — So certain integrals converge.

Consider S.E. (Integral Form) (in assymptotic limit)

$$\Psi(\underline{r}) = \Psi_0(\underline{r}) - \frac{m}{2\pi\hbar^2}\int_{\text{all space}} \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} V(\underline{r}')\Psi(\underline{r}')d\underline{r}'$$

$r >> r'$
So:

$$|\underline{r}-\underline{r}'| \simeq r - \hat{\underline{r}}\cdot\underline{r}' \simeq r$$

$$e^{ik(r-\hat{\underline{r}}\cdot\underline{r}')} = e^{ikr}e^{-i\underline{k}\cdot\underline{r}'}$$

if $\underline{k} = k\hat{\underline{r}}$

So:

$$\Psi(\underline{r}) \rightarrow e^{ikz} + \boxed{\frac{-m}{2\pi\hbar^2}\int_{\text{all space}} e^{-i\underline{k}\cdot\underline{r}'} V(\underline{r}')\Psi(\underline{r}')d^3\underline{r}'} \cdot \frac{e^{ikr}}{r}$$

Scattering amplitude → $f(\theta)$ (exact expression)

Now Say: Potential Weak ⇒ $\boxed{e^{ikz} \text{ is not altered much}}$

$$\Rightarrow \boxed{\Psi(\underline{r}') \simeq \Psi_0(\underline{r}')} \left(= e^{ikz'} \equiv e^{i\underline{k}'\cdot\underline{r}'} / \text{where } \underline{k}' = k\hat{\underline{z}}\right)$$

BORN APPROX

$$\Rightarrow \boxed{f(\theta,\phi) \equiv f(\underline{k},\underline{k}') = \frac{-m}{2\pi\hbar^2}\int_{\text{all space}} e^{i(\underline{k}'-\underline{k})\cdot\underline{r}'} V(\underline{r}')d^3\underline{r}'}$$

NB $\underline{k}'$ points along incident beam, $\underline{k}$ points to detector.

ie $f(\theta,\phi)$ is just Fourier Transform of V !

Simplifications: For spherically symmetric potential $V(\underline{r}) \equiv V(r)$

$$\boxed{f(\theta) \simeq -\frac{2m}{q\hbar^2}\int_0^\infty r V(r)\sin(qr)\,dr}$$

where $\underline{q} = \underline{k}'-\underline{k} = 2k\sin\left(\frac{\theta}{2}\right)$ and $\underline{q}\cdot\underline{r}' = qr'\cos\theta'$

For low energy scattering ie $\underline{k} \mathbin{/\!/} \underline{k}'$ (approx)

$$\boxed{f(\theta,\phi) \simeq \frac{-m}{2\pi\hbar^2}\int_{\text{all space}} V(\underline{r}')d^3\underline{r}'}$$

Lippmann-Schwinger E'qn

Let Ham $\hat{H} = \hat{H}_0 + \hat{V}$ where $H_0|\phi\rangle = \frac{p^2}{2m}|\phi\rangle$

free particle state

Scattering Potential

For ELASTIC scattering, eigenstate of H has same energy as that of H_0. ie if $H_0|\phi\rangle = E|\phi\rangle$

then $\boxed{H|\Psi\rangle = E|\Psi\rangle}$

but what is $|\Psi\rangle$? We know $|\Psi\rangle \rightarrow |\phi\rangle$ as $\hat{V} \rightarrow \hat{0}$

Solution: $\boxed{|\Psi\rangle = |\phi\rangle + \frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{V}|\Psi\rangle}$ LIPPMANN-SCHWINGER EQUATION

ε — infinitesimal to remove singularity - treat E as complex

In the position representation:

$$\boxed{\langle\underline{r}|\Psi\rangle = \langle\underline{r}|\phi\rangle + \int d\underline{r}'^3 \langle\underline{r}|\frac{1}{(E-\hat{H}_0+i\varepsilon)}|r'\rangle\langle r'|V|\Psi\rangle}$$

compare

so $G(\underline{r},\underline{r}') = \frac{\hbar^2}{2m}\langle\underline{r}|\frac{1}{(E-\hat{H}_0+i\varepsilon)}|\underline{r}'\rangle$

compare Ψ

and in assymp. limit, get $f(\underline{k},\underline{k}') = \frac{-m}{2\pi\hbar^2}\langle\underline{k}'|V|\underline{\Psi}\rangle$

Born Series - Transition Operator

Define $\hat{T}$ (transition op.) such that $\boxed{\hat{V}|\Psi\rangle = \hat{T}|\phi\rangle}$

So L-S. eq'n gives: $\hat{T}|\phi\rangle = \hat{V}|\phi\rangle + \hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}T|\phi\rangle$

Holds for all $|\phi\rangle$

So: $\hat{T} = \hat{V} + \hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{T}$ - operator equation.

Can substitute $\hat{T} \rightarrow \hat{T}$ to get iterative sol'n:

$$\hat{T} = \hat{V} + \left[\hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{V}\right] + \left[\hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{V}\right] + \ldots$$

it tells us how a disturbance propagates from one interaction to the next → Feynman Diagrams.

⟹ can write $f(\underline{k}',\underline{k}) = \frac{-m}{2\pi\hbar^2}\langle\underline{k}'|T|\underline{k}\rangle$

So if subst for T, get:

$$f(\underline{k}',\underline{k}) = \underbrace{\frac{-m}{2\pi\hbar^2}\langle\underline{k}'|\hat{V}|\underline{k}\rangle}_{\text{1st Born approx}} \underbrace{- \frac{m}{2\pi\hbar^2}\langle\underline{k}'|\hat{V}\frac{1}{(E-\hat{H}_0+i\varepsilon)}\hat{V}|\underline{k}\rangle}_{\text{2nd Born approx}} + \ldots$$

BORN SERIES ....

ie $f = \sum_{n=1}^{\infty} f^{(n)}$ (nth Born approximation)

Interpretation: $f^{(n)}$ is incident wave ($\underline{k}$) undergoing n sequential interactions before being scattered into direction $\underline{k}'$. Green f'n is called the Propagator

Geometrical Methods 1

Def'n of MANIFOLD:

A set M is a manifold if each pt of M has an $\boxed{\text{open neighbourhood}}$ which has a $\boxed{\text{continuous 1-1 map onto}}$ an open set of $\mathbb{R}^n$ for some n.

the set of all n-tuples: $[x_1, x_2 \dots x_n]$ where $x_i \in \mathbb{R}^1$ $i = 1 \dots n$.

distance function (not distance!)

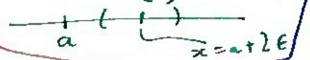
$$:= d(\underline{x}, y) = \left[(x_1-y_1)^2 + (x_2-y_2)^2 + \dots (x_n-y_n)^2\right]^{1/2}$$

$\underline{x}, y \in \mathbb{R}^n$

A neighbourhood is the points inside $d(\underline{x}, y)$ of $\underline{x}$ for eg...

A set is OPEN if every point in it has a neighbourhood entirely in it!

eg x s.t. $a < x < b$:



have induced a local topology on $\mathbb{R}^n$.

MAPPINGS

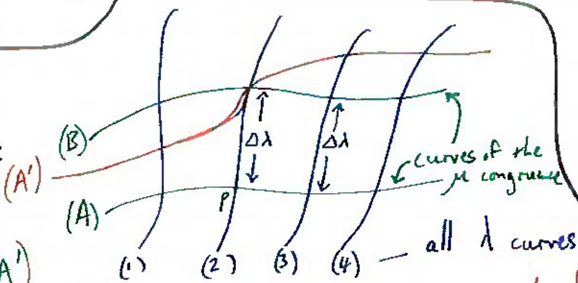
A map $f: M \to N$ associates a unique $f(x) = y \in N$ with an $x \in M$. (or $f: x \mapsto y$) The set of $f(x)$'s is the IMAGE of the x's. If a unique inverse image (ie set exists) then the map is 1-1. If $f: M \to N$ is from all of M then it is INTO N. If all of N has (an) image(s) → ONTO N

if only one image → 1-1

A map $f: M \to N$ is continuous at x in M if any open set of N containing $f(x)$, contains the image of an open set of M containing x.

Lie Derivatives:

Lie Dragging a vector field:



A' has been dragged from A

ie if Lie drag curve A, get A')

then if A' = B in all of manifold we have Lie dragged the vector field $\left(\frac{d}{d\mu}\right)$ using $\left(\frac{d}{d\lambda}\right)$

and $\left[\frac{d}{d\mu}, \frac{d}{d\lambda}\right] = 0$

ie $\left(\frac{d}{d\mu^{new}_{\Delta\lambda}} = \frac{d}{d\mu}\right)$

Lie Derivative of a scalar function:

ie take $f(\lambda_0 + \Delta\lambda)$, drag back to λ_0, minus $f(\lambda_0)$:

$$\pounds_{\bar{V}} f = \lim_{\Delta\lambda \to 0} \frac{f^*_{\Delta\lambda}(\lambda_0) - f(\lambda_0)}{\Delta\lambda}$$

but $f^*(\lambda_0) = f(\lambda_0 + \Delta\lambda)$

$$\Rightarrow \boxed{\pounds_{\bar{V}} f = \left(\frac{df}{d\lambda}\right)_{\lambda_0}}$$

of a vector field $\bar{u} = \frac{d}{d\mu}$:

Using arbitrary f'n, f: no

Result is: $\boxed{\pounds_{\bar{V}} \bar{u} = [\bar{V}, \bar{u}]}$

- coord. independent form of Partial derivative.

ie if $\bar{V} = \frac{\partial}{\partial x^1}$ (coord basis) for eg.

then $\left(\pounds_{\bar{V}} \bar{w}\right)^i = \frac{\partial W^i}{\partial x^1}$

of a one-form field $\tilde{\omega}(\ )$

if put vector field $\bar{w}$ get function!

$$\boxed{\pounds_{\bar{V}}\left[\tilde{\omega}(\bar{w})\right] = \left(\pounds_{\bar{V}}\tilde{\omega}\right)\bar{w} + \tilde{\omega}\left(\pounds_{\bar{V}}\bar{w}\right)}$$

(Liebniz Product Rule)

Invariance

A Tensor field is invariant if $\boxed{\pounds_{\bar{V}} T = 0}$

The vector space consisting of all vector fields under which T is invariant is a Lie Algebra!

These vector fields if T is the metric are called: KILLING VECTORS

ie Killing Vector is s.t. $\pounds_{\bar{V}} g = 0$

If $\bar{V} = \frac{\partial}{\partial x^1}$ ie a coord basis vector,

then $\frac{\partial}{\partial x^1} g_{ij} = 0$ - have got a Killing Vector!

$\boxed{\text{Conversely, if } g \text{ is indep of } x^i \text{ then } \frac{\partial}{\partial x^i} \text{ is a K.V.}}$

In particle dynamics - symm in pot → invariance but not for all symmetries. Reason: motion (with respect to which pot. is symmetric) must be along Killing Vector Field:

- eq'n of motion $m\dot{v}$ = (vector gradient of ϕ) $= g^{ij}\frac{\partial}{\partial x^j}\phi$ METRIC IS INVOLVED.

Example of Invariance: Axial Symmetry

If have an axially sym. system, get linear equation for solution: $\boxed{L(\Psi) = 0}$ where L is operator and $L(\phi) = L(\phi + \text{const})$

This doesn't mean that $\Psi(\phi) = \Psi(\phi + \text{const})$ but: can Fourier Analyse:

$$\Psi(\phi, x^j) = \sum_{m=-\infty}^{\infty} \Psi_m(x^j)\, e^{im\phi}$$

$\left(\begin{array}{l} j = r, \theta \\ \text{or } r, z \text{ etc} \end{array}\right)$

Then $\Psi_m(x^j)$ satisfy:

$$L_m(\Psi_m) = 0 \text{ where } L_m \text{ is given by:}$$

$$e^{-im\phi} L\left(\Psi_m e^{im\phi}\right) = 0$$

For eg:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

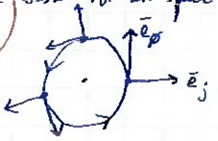
this is const when $\phi \to \phi + \text{const}$.

When it operates on $f(r,\theta)e^{im\phi}$ get $-m^2$ instead of $\frac{\partial^2}{\partial\phi^2}$

- this is L_m (not involving $\frac{\partial}{\partial\phi}$'s)

Ψ has axial eigenvalue m if $\pounds_{\bar{e}_\phi}\Psi = im\Psi$ cf.

This was OK. for scalar Ψ. But if dealing with eg. $\underline{A}$ (e/mag) then need vector axial harmonics ($e^{im\phi}$). Form basis for all space by Lie dragging $\bar{e}_\phi$ and $\bar{e}_j$ around $\bar{e}_\phi$:



The resulting field satisfies: $\boxed{\pounds_{\bar{e}_\phi}\bar{e}_j = 0}$

Lie Drag. condition.

Now have basis with axial e-val of zero:

So a basis with axial e-val m is $\left\{\bar{e}_{(m)j} = \bar{e}_j e^{im\phi},\ \bar{e}_{(m)\phi} = \bar{e}_\phi e^{im\phi}\right\}$

So now any vector that is an axial e-function of $\pounds_{\bar{e}_\phi}$ with eigenvalue m

ie $\pounds_{\bar{e}_\phi}\bar{V} = im\bar{V}$

can be written as a lin. combination:

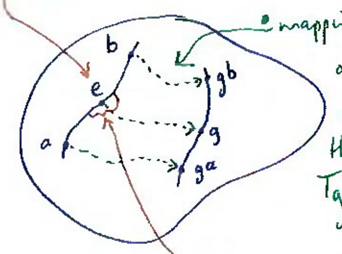
$$\boxed{\sum_j V^j \bar{e}_j e^{im\phi} + V^\phi \bar{e}_\phi e^{im\phi}}$$

Lie Groups:

Abstract Lie Groups

identity element

A Lie Group is a C^∞ manifold, G.



• mapping by Left Translation of G into G, carries elements of T_e (curves through e) into elements of T_g ie. $\boxed{L_g : T_e \to T_g}$

$L_g : G \to G$

Left trans. map.

(no) group elements "near" e

as below but for g

Tangent vectors at e. They form an n-dim. vector space.

• Left Invariant Vector Field on G is one such that $L_g : \bar V(\bar e) \to \bar V(g) \, (\forall g)$

"tangents to" ie remains the same $\bar V$

• Each ~~left~~ vector in T_e defines a left invariant vector field.

• If $\bar V$ and $\bar W$ are l.i.v.f's then so is $[\bar V, \bar W]$ ie l.i.v.f.'s form a Lie Algebra

• can express Lie Algebra using the Structure constants (components of the structure tensor):

$$\boxed{[\bar V_{(k)}, \bar V_{(l)}] = c^j_{kl} V_{(j)}}$$

• Consider integral curve (whose tangent vector (left inv.) is $\bar V$) at e:

$e \longrightarrow \bar V(e)$

- Can parametrise so that $t=0$ corresponds to e : $\bar V(t=0) = \bar V(e)$

Then − any point on curve (elements of G) can be generated by exponentiation of $\bar V$:

$$\boxed{g_{\bar V_e}(t) = e^{t\bar V}\Big|_e}$$ ie $g_V(\Delta t)$.... no.

• If have same path but diff. parameters,

$$g_{\bar V_e}(t_1)\, g_{\bar V_e}(t_2) = e^{(t_1+t_2)\bar V}\Big|_e$$

$$= g_{\bar V_e}(t_1+t_2)$$

So have a one parameter subgroup of G − it is abelian always (think)

• The "one param. subgroups of G" and the "Lie Alegebra Elements" are in 1-1 correspondence. (Isomorphic)

Example 1

$GL(n, R)$

(not "vectors" actually matrices...)

General linear group in n real dimensions. Any matrix A_e is a member of T_e and can therefore generate a one parameter subgroup and a left invariant vector field and an element of the Lie Algebra.

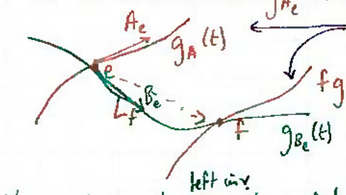
remember: one param. subgroup is the curve through e with tangent vector A_e (ie. the element of a left invariant field at e). So exponentiation of A generates the curve $g_A(t_0+t)$: (t_0 at e)

$$g_{A_e}(t_0+t) = g_{A_e}(t_0)\, g_{A_e}(t)$$

$$\Rightarrow \frac{dg_{A_e}}{dt} = g_{A_e}(t_0).A_e$$

$$\Rightarrow \boxed{g_{A_e} = e^{tA_e}} \, ?$$

Given: $\bar A_e$, $g_{A_e}(t)$, f (in G). Can use f to left-translate $g_{A_e}(t)$:



They are both curves of the congruence of the Left invariant vector field generated by A_e (and f) called $\bar A$

• Can put A into BLOCK DIAGONAL or CANONICAL form (like diagonalisation) by $A \to B^{-1}AB$. This also puts $e^{tB^{-1}AB}$ into canonical form.

• Not every element of $GL(n,R)$ is in a one-parameter subgroup − it is a disconnected grp.

Now − have two left inv. vector fields $\bar A$ and $\bar B$ generated by A_e and B_e. They are tangents to the integral curves g_A and g_B. The Lie bracket of $\bar A$ and $\bar B|_{\text{at } e}$ is [by using $g_A = \exp(tA_e)$] equal to the commutator of the generators A_e and B_e. $\bar A$ and $\bar B$, and thus $e^{([A_e, B_e]t)} = g_{[A_e,B_e]}(t)$ are all elements of the Lie Algebra of G. left invariant vector fields.

Example 2 $O(n)$ − The Euclidean Symmetry Group.

Orthogonal because: $O^T I O = I$. Every element of $O(n)$ has determinant ± 1. Those with +1 form $SO(n)$ → rotations

Those with −1 : Inversions. They do not form a group (Think!) [change handedness of a ~~xx~~ basis]

The Lie Algebra of $O(n)$ is {all antisym. matrices}

$O(n)$ is a subgroup of $GL(n, R)$

a basis is : $L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $L_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Hey ~~xx~~! $[L_i, L_j] = \varepsilon_{ijk} L_k$

↑ structure constants.

Example 3 $SU(n)$ − A subgroup of $GL(n, C)$

Analogous to $SO(n)$ from above:

replace Orthogonal with Unitary

Antisymmetric with Antihermitian

− its Lie Algebra is Antihermitian with zero trace.

A basis for T_e is: $J_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $J_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

and $[J_i, J_j] = \varepsilon_{ijk} J_k$

Same structure tensor ∴ same Lie Algebra

⇒ connection between SO3, SU2.

Actually it is a Homomorphism....

Lie Algebras

Formal Definition: A Lie Algebra is a real vector space on which ∃ a bilinear rule of combination [,] which obeys:

① $[\bar A, \bar B] = -[\bar B, \bar A]$

② Jacobi Identity.

Ordinary Euclidean vector product is such a rule and $\bar a_i \times \bar a_j = \varepsilon_{ijk} \bar a_k$

Very Important Theorem:

• Every Lie Alg. is the Lie Algebra of one and only one simply-connected Lie Group.

And • any other Lie Group with the same Lie Algebra is covered by the simply connected one.

To see if a group is simply connected or not, need to find a diffeomorphism (C^∞ 1-1 map) onto a simply connected manifold.

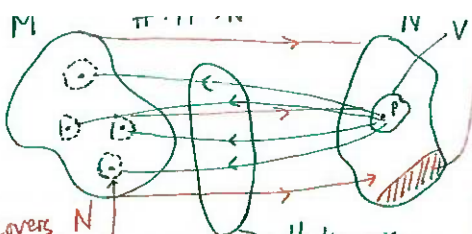
eg SU2 is diffeomorphic onto the 3-sphere. NOT our normal sphere (2-sphere)

Lie Groups (cont'd)

Lie Algebras (cont'd)

Covering: A manifold M covers another N if there is a map π of M onto N such that the inverse image of P (in N)'s neighbourhood V is a disjoint union of open neighbourhoods of the points $\pi^{-1}(P)$ in M. In pictures:

(simply connected)

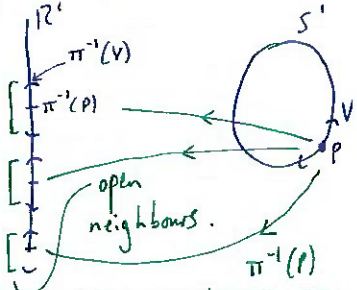


all 4 maps are $\pi^{-1}(P)$

M covers N because these open neighbourhoods disjointly unioned form the inverse image of V (the neighbourhood in N)

remove bits of N that don't have inverse images then the map π is onto.

Example: $\mathbb{R}^1$ covers S^1:



Example: SU(2) covers SO(3) multiply...

To find π, construct the one-parameter subgroups of J_1 (in SU(2)) and L_1 (in SO(3)) through exponentiation of J_1, L_1 (generators)

$$e^{tJ_1} = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) \\ i\sin(t/2) & \cos\left(\frac{t}{2}\right) \end{pmatrix} \qquad e^{tL_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

A map could be: $\boxed{\pi : e^{tJ_1} \mapsto e^{tL_1}}$ A homomorphism (of the two one-param subgroups.)

Now: $(t + 4n\pi)$ for $n \in \mathbb{Z}$ in SU(2) maps to the same element of SO(3)

and $t + 4n\pi$ is two points only of SU(2)

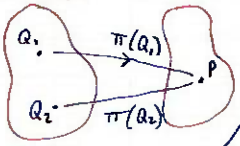
If generalise to whole basis then:

$$\boxed{\pi : e^{(t_1J_1 + t_2J_2 + t_3J_3)} \longmapsto e^{(t_1L_1 + t_2L_2 + t_3L_3)}}$$

is a double covering of SO(3) by SU(2)
– A HOMOMORPHISM.

So a single covering is an isomorphism...

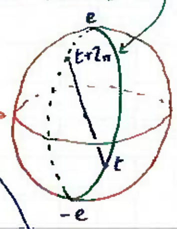
Picture:



GLOBAL TOPOLOGY:

Know that SU(2) has global topology of the 3-sphere - can deduce that of SO(3).

A 2-sphere is a 2 slice of S^3. It can contain the one-parameter subgroup e^{tJ_1} as a great circle - begins at e (t=0) and returns to e (t=4π):



To make both pts shown (t and t+2π) into the same pt. in SO(3), we identify the top half of the sphere with the parameter space of SO(3). Then identify equatorial pts sep. by a diameter with each other

So a curve involving any point on the equator cannot be shrunk to a point as the diameter is fixed! ⇒ not simply connected!

Also: $\mathbb{R}^3$ is covered by SU(2) [same Lie Algebra] but is more common to associate vectors in $\mathbb{R}^3$ to rotations (elements of SO(3)) ie $e^{\theta L_1}$ assoc. with $e^{\theta \bar{e}_1}$

(rot. of θ about x axis) (vector, length θ along x axis)

However, $\mathbb{R}^3$ and SO(3) having an equal number of dimensions is purely coincidental. Eg SO(4) is 6-d but it acts in $\mathbb{R}^4$ which is 4d. So:
- can associate vector in $\mathbb{R}^3$ with element of SU(2) equally well! ∴ can associate spin of e^- with vector even though "spin" does not exist in the tangent space of $\mathbb{R}^3$

Realizations and Representations

If we associate with each element $g \in G$ a transformation T(g) of some space M that preserves the group properties then we have made a realisation of the group.

esp: $T(g_2) \circ T(g_1) = T(g_2 g_1)$ Group Closure Property.

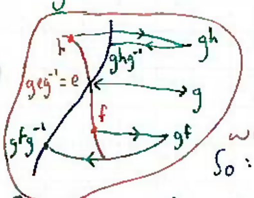
If M is a VECTOR space and every T(g) is a linear trans. (ie. a (1/1) tensor) then the realisation is called a representation. If 1-1, faithful.

Have actually introduced SO(3) as matrices but having done so, should back up and regard as abstract elements (obeying axioms...). Then may find other useful representations of the same group.

The adjoint representation of any group is as linear transformations of the vector space of its own Lie Algebra!

Consider $I_g : h \mapsto ghg^{-1}$.

This maps one-param subgroups into other one-param. subgroups and tangent vectors in T_e into other tangent vectors in T_e:



Study very carefully

Let red one-param subgroup be $e^{t\bar{X}_e}$

The adjoint-transformation of X into the blue curve's tangent at e is written as $Ad_g : \bar{X} \mapsto$ (blue x).

So: $I_g[e^{t\bar{X}}] = e^{t\,Ad_g(\bar{X})}$

If g itself is a one parameter subgroup say $e^{s\bar{Y}}$ then $Ad_{g(s)}[\bar{X}] = e^{(s\mathcal{L}_{\bar{Y}})}\bar{X}$

Representations of SO(3)

When elements of SO(3) act on E^3, can write them as the rotation matrices. But $L^2(S^2)$ is another vector space and SO(3) has another representation in it – as differential operators. $L^2(S^2)$ is the space of all complex square-integrable functions on the 2-sphere ie with coords (θ, ϕ). $L^2(S^2)$ has ∞ dimension

an irrep

So given any old basis $\{f_i\}$, $R(g)[a^i f_i] = b^j f_j$ (rep of rot)

but $\underbrace{\int f_i R(g) f_j \, d\mathrm{sph}_j}\, a^i = b^j$ ie $R_i^{\ j} a^i = b^j$ (matrix element)

$$\text{So}: \begin{pmatrix} R_{11} & R_{21} & \cdots & \infty \\ R_{12} & \ddots & & \\ \vdots & & \ddots & \\ \infty & & & \infty \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \infty \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \infty \end{pmatrix}$$

P.T.O. very quickly.

Representations of SO(3) (cont'd)

If choose a cunning basis, Y_{lm} then matrix equation (and the vector space $L^2(S^2)$) splits into invariant vector subspaces:

$-\infty < l \leq \infty$
$-l \leq m \leq l$
so together, go $-\infty \to \infty$
$\therefore$ spans all of $L^2(S^2)$

$$(V^1)$$

$$\begin{pmatrix} (R_{00}=0)^{l=0} & 0 \quad 0 \quad 0 & & \\ \begin{matrix}0\\0\\0\end{matrix} & \begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}_{l=1} & 0 & \\ & 0 & l=2\begin{pmatrix}-2&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&2\end{pmatrix} & \ddots \end{pmatrix} \begin{pmatrix} (Y_{00})^{l=0} \\ \begin{pmatrix}Y_{1-1}\\Y_{10}\\Y_{11}\end{pmatrix} \\ \begin{pmatrix}Y_{2-2}\\Y_{2-1}\\Y_{20}\\\vdots\end{pmatrix} \end{pmatrix} = \begin{pmatrix} (Y_{00}) \\ \begin{pmatrix}Y_{1-1}\\Y_{10}\\Y_{11}\end{pmatrix}_{l=1} \\ \begin{pmatrix}Y_{2-2}\\Y_{2-1}\\Y_{20}\\Y\\\vdots\end{pmatrix}_{l=2} \end{pmatrix}$$

Each V^l is associated with an irreducible representation of SO(3), with matrix elements in $f_{mm'}$.

For $l=1$ (ie V^1), the invariant vector subspace has the same dimension as E^3 and $\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}$ is nothing but the z-axis rotation matrix $\begin{pmatrix}\cos\phi & -s\phi & 0\\ s\phi & c\phi & 0\\ 0 & 0 & 1\end{pmatrix}$ except refered to the spherical (harm) basis set Y_{1-1}, Y_{10}, Y_{11} instead of x, y and z !!

$\boxed{Y_{1-1} \propto x+iy}$ $\boxed{Y_{11} \propto x-iy}$

and $\boxed{Y_{10} \propto z}$